**Pset 11 - Part I**

(Part II will be available by 11/27.)

Due December 3 (4 points each)

We first present the following definitions, given for any sequence $\{a_n\}$:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \big( \inf_{m\geq n} a_m \big)$$

and

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \big( \sup_{m\geq n} a_m \big).$$

To give you some intuition, we determine the $\liminf, \limsup$ for a few sequences. (You may want to draw some pictures or write out some terms of the sequence to help yourself.)

- Let $a_n = (-1)^n \big(\frac{n-1}{n}\big)$. Then $\liminf a_n = -1$ and $\limsup a_n = 1$. Notice that $\lim_{n\to\infty} a_n$ does not exist.
- Let
$$b_n = \begin{cases} 0 & : n \textit{ even} \\ \frac{1}{n} & : n \textit{ odd} \end{cases}$$
Then $\liminf b_n = \limsup b_n = \lim b_n = 0$.

**Problems**

(1) Prove a sequence converges if and only if its $\liminf$ equals its $\limsup$.

(2) Use this fact to prove every Cauchy sequence of real numbers converges. (You will find the definition of a Cauchy sequence on the third practice exam.)

(3) Carefully prove that $a_n \to 0$ is a necessary condition for $\sum a_n$ to converge. (Be more clear and thorough than my outline from class.)

(4) A function $f$ on $\mathbb{R}$ is compactly supported if there exists a constant $B > 0$ such that $f(x) = 0$ if $|x| \geq B$. If $f$ and $g$ are two differentiable, compactly supported functions on $\mathbb{R}$, then we define
$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$
Note we define $\int_{-\infty}^{\infty} f(x)dx = \lim_{t\to\infty} \int_{-t}^{0} f(x)dx + \lim_{t\to\infty} \int_{0}^{t} f(x)dx$.
    - Prove $(f * g)(x) = (g * f)(x)$.
    - Prove $(f' * g)(x) = (g' * f)(x)$.