18.014 Problem Set 11 Solutions

Total: 32 points

Problem 1: Prove that a sequence converges if, and only if its liminf equals its limsup.

Solution (4 points) Suppose $\{a_n\}$ is a sequence that converges to a limit L. Then given $\epsilon > 0$, there exists an integer N such that $n > N$ implies $|a_n - L| < \frac{\epsilon}{2}$. In particular, the set $\{a_n\}$ for $n > N$ is bounded below by $L - \frac{\epsilon}{2}$ and bounded above by $L + \frac{\epsilon}{2}$. Thus,

$$L + \frac{\epsilon}{2} \geq \inf_{n \geq N+1} a_n \geq L - \frac{\epsilon}{2} \text{ and } L + \frac{\epsilon}{2} \geq \sup_{n \geq N+1} a_n \geq L - \frac{\epsilon}{2}.$$

In particular, given $\epsilon > 0$, there exists N such that $m > N$ implies

$$\left| \inf_{n \geq m} a_n - L \right| < \epsilon, \ \left| \sup_{n \geq m} a_n - L \right| < \epsilon.$$

We conclude $\liminf a_n = L = \limsup a_n$ and the liminf equals the limsup.

Conversely, suppose $\{a_n\}$ is a sequence such that $\liminf a_n = L = \limsup a_n$. We will show $\lim_{n\to\infty} a_n = L$. Given $\epsilon > 0$, there exist N_1 and N_2 such that $m > N_1$ implies

$$\left| \inf_{n \geq m} a_n - L \right| < \epsilon$$

and $m > N_2$ implies

$$\left| \sup_{n \geq m} a_n - L \right| < \epsilon.$$

Let $N = \max\{N_1, N_2\}$. If $m > N_1, N_2$, then

$$L - \epsilon < \inf_{n \geq m} a_n \leq a_m \leq \sup_{n \geq m} a_n < L + \epsilon.$$

In particular,

$$|a_m - L| < \epsilon$$

if $m > N$. Thus, $\lim_{n\to\infty} a_n = L$.

Problem 2: Use this fact to prove every Cauchy sequence of real numbers converges.

Solution (4 points) First, recall the following lemma:

Lemma: Every decreasing sequence that is bounded below converges, and every increasing sequence that is bounded above converges.

Now, let $\{a_n\}$ be a Cauchy sequence of real numbers. Then there exists M such that $n, m > M$ implies $|a_n - a_m| < 1$. Putting $m = M + 1$, we observe $a_{M+1} - 1 \leq a_n \leq a_{M+1} + 1$ if $n > M$. Put $C = \max\{a_{M+1}, a_1, \ldots, a_M\}$ and put $B = \min\{a_{M+1} - 1, a_1, \ldots, a_M\}$. Then

$$B \leq a_n \leq C$$

for all n. In particular, if $b_n = \sup_{m \geq n} a_m$ and $c_n = \inf_{m \geq n} a_m$, then $\{b_n\}$ is a decreasing sequence bounded below by B and $\{c_n\}$ is an increasing sequence bounded above by C. Thus, by the lemma $\{b_n\}$ converges to L_1 and $\{c_n\}$ converges to L_2.

To finish the proof, we again use that $\{a_n\}$ is a Cauchy sequence. Given $\epsilon > 0$, there must exist N such that $n, m > N$ implies $|a_n - a_m| < \frac{\epsilon}{5}$. Moreover, there must exist $M_1 > N$ such that $|b_{M_1} - L_1| < \frac{\epsilon}{5}$ and $M_2 > N$ such that $|c_{M_2} - L_2| < \frac{\epsilon}{5}$. We can do this because $\lim_{n\to\infty} b_n = L_1$ and $\lim_{n\to\infty} c_n = L_2$. Choose $n \geq M_1$ such that $|a_n - b_{M_1}| < \frac{\epsilon}{5}$ and choose $m \geq M_2$ such that $|a_m - c_{M_2}| < \frac{\epsilon}{5}$. We can do this because $b_{M_1} = \sup_{k \geq M_1} a_k$ and $c_{M_2} = \inf_{k \geq M_2} a_k$. Now, we observe

$$|L_1 - L_2| \leq |L_1 - b_{M_1}| + |b_{M_1} - a_n| + |a_n - a_m| + |a_m - c_{M_2}| + |c_{M_2} - L_2| < \epsilon.$$

Since this is true for every $\epsilon > 0$, we conclude that $L_1 = L_2$. But, by the previous problem, if the liminf and the limsup are equal, then $\lim_{n\to\infty} a_n$ exists. Therefore, every Cauchy sequence converges.

Problem 3: Suppose the series $\sum_{n=1}^{\infty} a_n$ converges. Then $\lim_{n\to\infty} a_n = 0$.

Solution (4 points) Define $s_m = \sum_{n=1}^{m} a_n$. Then by definition $\{s_m\}$ converges. In particular, $\{s_m\}$ is a Cauchy sequence (Problem 5 on the last practice exam). Thus, given $\epsilon > 0$, there exists N such that $m, n > N$ implies $|s_n - s_m| < \epsilon$. If we choose $n = m + 1$, then we get

$$|a_m| = |s_{m+1} - s_m| < \epsilon$$

whenever $m > N + 1$. We conclude $\lim_{m\to\infty} a_m = 0$.

Problem 4: A function f on $\mathbb{R}$ is compactly supported if there exists a constant $B > 0$ such that $f(x) = 0$ if $|x| \geq B$. If f and g are two differentiable, compactly supported functions on $\mathbb{R}$, then we define

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

Prove (i) $f * g = g * f$ and (ii) $f' * g = f * g'$.

Solution (4 points) a) Using the substitution $u = x - y$, we have

$$\int_{-t}^{t} f(x-y)g(y)dy = -\int_{x+t}^{x-t} f(u)g(x-u)du = \int_{x-t}^{x+t} f(u)g(x-u)du.$$

Using that f is compactly supported, choose B such that $f(u) = 0$ if $|u| > B$. Thus, if $t > B + |x|$, then

$$\int_{x-t}^{x+t} f(u)g(x-u)du = \int_{x-t}^{-B} f(u)g(x-u) + \int_{-B}^{B} f(u)g(x-u)du + \int_{B}^{x+t} f(u)g(x-u)du.$$

The first and third terms are zero since $f(u)$ is zero whenever $u < -B$ or $u > B$. Hence, our integral becomes

$$\int_{-B}^{B} f(u)g(x-u)du.$$

Similarly,

$$\int_{-t}^{t} g(x-u)f(u)du = \int_{-B}^{B} g(x-u)f(u)du$$

if $t > B$. And we have

$$(f * g)(x) = \lim_{t\to\infty} \int_{x-t}^{x+t} f(u)g(x-u)du = \int_{-B}^{B} f(u)g(x-u)du$$

$$= \lim_{t\to\infty} \int_{-t}^{t} g(x-u)f(u)du = (g * f)(x).$$

b) Integration by parts tells us

$$\int_{-t}^{t} f'(x-y)g(y)dy = -f(x-y)g(y)\Big|_{-t}^{t} + \int_{-t}^{t} f(x-y)g'(y)dy.$$

The limit of the first term on the right as $t \to \infty$ is

$$\lim_{t\to\infty}(-f(x-t)g(t)+f(x+t)g(-t))=0$$

since $g(t)=0$ and $g(-t)=0$ if $t>B'$ for some $B'>0$. Thus,

$$(f'*g)(x)=\lim_{t\to\infty}\int_{-t}^{t}f'(x-y)g(y)dy=\lim_{t\to\infty}\int_{-t}^{t}f(x-y)g'(y)dy=(f*g')(x).$$

Applying part (a), we get $(f'*g)(x)=(f*g')(x)=(g'*f)(x)$ as desired.

Problem 5: Determine whether the series diverge, converge conditionally, or converge absolutely.
(a) $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+100}$ (b) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{2n+100}{3n+1}\right)^n$.

Solution (4 points) (a) Consider the function $f(x)=\frac{\sqrt{x}}{x+100}$. Note

$$f'(x)=\frac{1}{2\sqrt{x}(x+100)}-\frac{\sqrt{x}}{(x+100)^2}.$$

One observes $f'(x)<0$ if $x>100$. Hence, f is monotonically decreasing when $x>100$. Moroever, its easy to see $\lim_{x\to\infty}f(x)=0$. Now, we break up our sum into

$$\sum_{n=1}^{100}(-1)^n\frac{\sqrt{n}}{n+100}+\sum_{n=101}^{\infty}(-1)^n\frac{\sqrt{n}}{n+100}.$$

The first term is a finite sum and the second term converges by Leibniz's rule (Thm. 10.14). Thus, our series converges.
However, our series does not converge absolutely. To see this, let $a_n=\frac{\sqrt{n}}{n+100}$, $b_n=\frac{1}{\sqrt{x}}$, and note $\lim_{n\to\infty}\frac{a_n}{b_n}=1$. By example one on page 398, we know that $\sum b_n$ diverges. Hence, by theorem 10.9, $\sum a_n$ diverges as well.

(b) This sum converges absolutely. Let $a_n=\left(\frac{2n+100}{3n+1}\right)^n$ and $b_n=\left(\frac{2}{3}\right)^n$. Observe $\lim_{n\to\infty}\frac{a_n}{b_n}=1$ and $\sum b_n$ converges since it is a geometric series. Hence, by theorem 10.9, $\sum a_n$ converges as well.

Problem 6: Prove $\sum_{n=1}^{\infty}a_n$ converges absolutely if $a_n=1/n$ if n is a square and $a_n=1/n^2$ otherwise.

Solution (4 points) Let $s_N = \sum_{n=1}^{N} a_n$ be the nth partial sum. Note

$$s_N = \sum_{\substack{n\le N \\ \text{not a square}}} \frac{1}{n^2} + \sum_{m\le\sqrt{N}} \frac{1}{m^2} \le \sum_{n\le N} \frac{2}{n^2}.$$

But, $\sum_{n\le N} \frac{2}{n^2} \le 2\sum_{n=1}^{\infty} \frac{1}{n^2}$. This is a finite number, C, by example one on page 398. Since the partial sums s_N are an increasing sequence, bounded by C, they must converge by our lemma in problem 2.

Problem 7: (a) Prove that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges. Give a counterexample in which $\sum_{n=1}^{\infty} a_n^2$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

(b) Find all real c for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^c}{(3n)!}$ converges.

Solution (4 points) (a) Suppose $\sum_{n=1}^{\infty} |a_n|$ converges. By problem 3, we must have $\lim_{n\to\infty} |a_n| = 0$. Thus, there exists N such that $|a_n| < 1$ whenever $n > N$. In particular, we see $a_n^2 = |a_n|^2 < |a_n|$ if $n > N$. Splitting up our series, we have

$$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{N} a_n^2 + \sum_{n=N+1}^{\infty} a_n^2.$$

The first sum is finite because it is a finite sum. We may compare the second series term by term to $\sum_{n=N+1}^{\infty} |a_n|$, which converges by hypothesis.
On the other hand, $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2$ converges by example one on the top of page 398. Yet, $\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series.

(b) Let $b_n = \frac{(n!)^c}{(3n)!}$. First, we apply the ratio test, and we get

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{(n+1)^c}{(3n+3)(3n+2)(3n+1)}.$$

This limit is zero and the series converges if $c < 3$. The limit is ∞ and the series diverges if $c > 3$. For $c = 3$, we analyze each term. Note

$$\frac{(n!)^3}{(3n)!} = 1 \cdot \prod_{k=1}^{n} \frac{k}{n+k} \prod_{k=1}^{n} \frac{k}{2n+k} \le \frac{1}{2^n}$$

since $2k \le n+k$ and $k \le 2n+k$. But, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges because it is a geometric series. Thus, by the comparison test (Thm. 10.8), we conclude that our series converges when $c = 3$.

Problem 8: (a) Prove that $\lim_{n\to\infty} \sum_{k=qn}^{pn} \frac{1}{k} = \log(p/q)$. (b) Show the series $1 + 1/3 + 1/5 - 1/2 - 1/4 + 1/7 + 1/9 + 1/11 - 1/6 - 1/8 \ldots$ converges to $\log 2 + \frac{1}{2}\log(3/2)$.

Solution (4 points) (a) We let $\epsilon > 0$. First choose N such that $\frac{1}{pn} < \epsilon/2$ for all $n \geq N$. Then for all $n \geq N$,

$$\left(\sum_{k=qn}^{pn} \frac{1}{k} - \sum_{k=qn}^{pn-1} \frac{1}{k} \right) = \frac{1}{pn} < \epsilon/2.$$

Now, choose M such that $\frac{p-q}{pqn} < \epsilon/2$ for all $n \geq M$. As the function $f(x) = 1/x$ is monotonically decreasing, we get the estimate

$$\left(\sum_{k=qn}^{pn-1} \frac{1}{k} - \int_{qn}^{pn} \frac{dx}{x} \right) \leq \frac{1}{qn} - \frac{1}{pn} = \frac{p-q}{npq} < \epsilon/2.$$

Now, choose $\tilde{N} = \max N, M$ and observe $\int_{qn}^{pn} \frac{dx}{x} = \log(x)|_{qn}^{pn} = \log(p/q)$. Thus, for all $n \geq \tilde{N}$, the triangle inequality and our work above implies:

$$\left| \sum_{k=qn}^{pn} \frac{1}{k} - \log(p/q) \right| \leq \left| \sum_{k=qn}^{pn} \frac{1}{k} - \sum_{k=qn}^{pn-1} \frac{1}{k} \right| + \left| \sum_{k=qn}^{pn-1} \frac{1}{k} - \log(p/q) \right| < \epsilon/2 + \epsilon/2 = \epsilon.$$

(b) We begin by observing that

$$s_{5m} = \sum_{k=1}^{3m} \frac{1}{2k-1} - \sum_{k=1}^{2m} \frac{1}{2m}.$$

Now,

$$\sum_{k=1}^{3m} \frac{1}{2k-1} = \sum_{k=1}^{6m} \frac{1}{k} - \sum_{k=1}^{3m} \frac{1}{2k} = \sum_{k=1}^{6m} \frac{1}{k} - \sum_{k=1}^{3m} \frac{1}{k} + \sum_{k=1}^{3m} \frac{1}{2k}$$

and thus

$$s_{5m} = \sum_{k=3m+1}^{6m} \frac{1}{k} + \frac{1}{2} \sum_{k=2m+1}^{3m} \frac{1}{k} = \sum_{k=3m}^{6m} \frac{1}{k} + \frac{1}{2} \sum_{k=2m}^{3m} \frac{1}{k} + \left(\frac{1}{3m} + \frac{1}{2m} \right).$$

Thus $\lim_{m\to\infty} s_{5m} = \log(6m/3m) + \frac{1}{2}\log(3m/2m) = \log 2 + \frac{1}{2}\log(3/2)$.