18.014 Problem Set 1 Solutions

Total: 24 points

Problem 1: If $ab = 0$, then $a = 0$ or $b = 0$.

Solution (4 points)
Suppose $ab = 0$ and $b \neq 0$. By axiom 6, there exists a real number y such that $by = 1$. Hence, we have

$$a = 1 \cdot a = a \cdot 1 = a(by) = (ab)y = 0 \cdot y = 0$$

using axiom 4, axiom 1, axiom 2, and Thm. I.6. We conclude that a and b cannot both be non-zero; thus, $a = 0$ or $b = 0$.

Problem 2: If $a < c$ and $b < d$, then $a + b < c + d$.

Solution (4 points) By Theorem I.18, $a + b < c + b$ and $b + c < d + c$. By the commutative axiom for addition, we know that $c+b = b+c, d+c = c+d$. Therefore, $a + b < c + b, c + b < c + d$. By Theorem I.17, $a + b < c + d$.

Problem 3: For all real numbers x and y, $\big||x| - |y|\big| \leq |x - y|$.

Solution (4 points)
By part (i) of this exercise, $|x|-|y| \leq |x-y|$. Now notice that $-(|x|-|y|) = |y|-|x|$. By definition of the absolute value, either $||x|-|y|| = |x|-|y|$ or $||x|-|y|| = |y|-|x|$. In the first case, by part (i) of this problem, we see that $||x| - |y|| \leq |x - y|$. In the second case, we can interchange the x and y from part (i) to get $||x| - |y|| = |y| - |x| \leq |y - x| = |x - y|$, where the last equality comes from part (c) of this problem. Thus, $||x| - |y|| \leq |x - y|$.

Problem 4: Let P be the set of positive integers. If $n, m \in P$, then $nm \in P$.

Solution (4 points)

Fix $n \in P$. We show by induction on m that $nm \in P$ for all $m \in P$.
First, we check the base case. If $m = 1$, then

$$nm = n \cdot 1 = 1 \cdot n = n \in P$$

by axiom 4, axiom 1, and the hypothesis $n \in P$.
Next, we assume the statement for $m = k$ and we prove it for $m = k + 1$. Assume $nk \in P$. By theorem 5 of the course notes, $nk+n \in P$. By axiom 3, $nk+n = n(k+1)$; thus, $n(k + 1) \in P$ and our induction is complete.

Problem 5: Let $a, b \in \mathbb{R}$ be real numbers and let $n \in P$ be a positive integer. Then $a^n \cdot b^n = (a \cdot b)^n$.

Solution (4 points)
Fix $a, b \in \mathbb{R}$. We prove the statement by induction on n.
First, we must check the statement for $n = 1$. In that case, we must show $a^1 \cdot b^1 = (a \cdot b)^1$. By the definition of exponents, we know $a^1 = a$, $b^1 = b$, and $(a \cdot b)^1 = a \cdot b$ so our statement becomes the tautology $a \cdot b = a \cdot b$.
Next, we check the inductive step. Assume the statement is true for $n = k$; we must prove it for $n = k + 1$.
Notice that $(ab)^{k+1} = (ab)^k \cdot (ab)^1 = a^k \cdot b^k \cdot a^1 \cdot b^1$ by Theorem 10 from the course notes and the induction hypothesis. As $a^k \cdot b^k \cdot a^1 \cdot b^1 = a^k \cdot a^1 \cdot b^k \cdot b^1 = a^{k+1} \cdot b^{k+1}$ by commutativity and Theorem 10, we see that the statement holds for $n = k + 1$.

Problem 6: Let a and h be real numbers, and let m be a positive integer. Show by induction that if a and $a + h$ are positive, then $(a + h)^m \geq a^m + ma^{m-1}h$.

Solution (4 points)
The first step is to prove the statement for $m = 1$. In this case $(a+h)^m = (a+h)^1 = a + h$ by the definition of exponents and

$$a^m + ma^{m-1}h = a^1 + 1 \cdot a^{(1-1)}h = a + a^0h = a + 1 \cdot h = a + h$$

where the second to last inequality used the definition $a^0 = 1$. Hence, for $m = 1$, we have $(a+h)^m = a^m + ma^{m-1}h$, which in particular implies $(a+h)^m \leq a^m + ma^{m-1}h$ by the definition of $\leq$.
Next, we assume the statement for $m = k$, and then we prove it for $m = k + 1$. Thus, we assume $(a + h)^k \geq a^k + ka^{k-1}h$, which means $(a + h)^k = a^k + ka^{k-1}$ or

$(a+h)^k > a^k + ka^{k-1}h$. In the first case, we can multiply both sides by $(a+h)$ to get $(a+h)^k \cdot (a+h) = (a^k + ka^{k-1})(a+h)$. In the second case, we can use Thm. I.19 and the fact that $a+h > 0$ to conclude $(a+h)^k \cdot (a+h) > (a^k + ka^{k-1}h) \cdot (a+h)$. Thus, by the definition of exponents and the definition of $\geq$, we have

$$(a+h)^{k+1} = (a+h)^k \cdot (a+h) \geq (a^k + ka^{k-1}h) \cdot (a+h) = a^{k+1} + (k+1)a^k h + ka^{k-1}h^2.$$

To finish the proof we need a lemma.

Lemma: If a is a positive real number, then a^l is positive for all positive integers l.

The proof is by induction on l. When $l = 1$, we know $a^l = a^1 = a$ using the definition of exponents. However, a is positive by hypothesis so the base case is true.
Now we assume the result for l and prove it for $l+1$. Note $a^{l+1} = a^l \cdot a$ by the definition of exponents. Further, by the problem 5, a^{l+1} is positive since a^l is positive by the induction hypothesis and a is positive by the hypothesis of the lemma. The lemma follows.

Now, back to our proof. By the lemma, we know a^{k-1} is positive since $k-1$ is a positive integer and a is positive. Moreover, all positive integers are positive, as is remarked in the course notes; thus, ka^{k-1} is positive by problem 5. Now, if $h = 0$, then $h^2 = 0$ by Thm. I.6; hence, $(ka^{k-1})h^2 = 0$ again by Thm. I.6. Putting this together with the expression (*) above yields

$$(a+h)^{k+1} \geq a^{k+1} + (k+1)a^k h + ka^{k-1}h^2 = a^{k+1} + (k+1)a^k h.$$

On the other hand, if $h \neq 0$, then $h^2 > 0$ by Thm. I.20; hence, $ka^{k-1}h^2 > 0$ by Thm. I.19. Adding $a^{k+1} + (k+1)a^k h$ to both sides (using Thm. I.18) yields $a^{k+1} + (k+1)a^k h < a^{k+1} + (k+1)a^k h + ka^{k-1}h^2$. Combining this with (*) and applying the transitive property (Thm. I.1.7) implies $(a+h)^{k+1} > a^{k+1} + (k+1)a^k h$. In particular,

$$(a+h)^{k+1} \geq a^{k+1} + (k+1)a^k h.$$

Thus, regardless of whether $h = 0$ or $h \neq 0$, we have proved the statement for $m = k+1$. The claim follows.

Bonus: If $x_1, \ldots, x_n$ are positive real numbers, define

$$A_n = \frac{x_1 + \cdots + x_n}{n}, \; G_n = (x_1 \cdots x_n)^{1/n}.$$

(a) Prove that $G_n \leq A_n$ for $n = 2$.
(b) Use induction to show $G_n \leq A_n$ for any $n = 2^k$ where k is a positive integer.
(c) Now show $G_n \leq A_n$ for any positive integer n.

Solution (4 points)
Because this is a bonus problem, this solution is a bit less rigorous than the others. However, you should be able to fill in all of the details on your own.
(a) Note $(x_1 - x_2)^2 \geq 0$. Expanding, we get

$$(x_1 - x_2)^2 = x_1^2 + x_2^2 - 2x_1x_2 = (x_1 + x_2)^2 - 4x_1x_2$$

is also positive. Hence, $(x_1 + x_2)^2 \geq 4x_1x_2$. Dividing by 4 and taking square roots, we get

$$\frac{x_1 + x_2}{2} \geq (x_1x_2)^{1/2}.$$

(b) We prove this part by induction on k. The base case $k = 1$ was done in part a. Now, we assume $G_n \leq A_n$ for $n = 2^k$, and we prove it for $n = 2^{k+1}$. The inductive hypothesis tells us that

$$Y_1 = \frac{x_1 + \cdots + x_{2^k}}{2^k} \geq (x_1 \cdots x_{2^k})^{1/2^k}$$

and

$$Y_2 = \frac{x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^k} \geq (x_{2^k+1} \cdots x_{2^{k+1}})^{1/2^k}.$$

Using part (a), we know

$$\frac{Y_1 + Y_2}{2} \geq (Y_1Y_2)^{1/2}.$$

Writing this in terms of the x_i, we have

$$\frac{x_1 + \cdots + x_{2^{k+1}}}{2^{k+1}} \geq (x_1 \cdots x_{2^{k+1}})^{1/2^{k+1}}.$$

(c) Select a positive integer m such that $2^m > n$. Fix positive real numbers $x_1, \ldots, x_n$, and let

$$A_n = \frac{x_1 + \cdots + x_n}{n}.$$

Now, put $A_n = x_{n+1} = x_{n+1} = \cdots = x_{2^m}$. Applying part (b) for these real numbers $x_1, \ldots, x_{2^m}$ yields

$$\frac{x_1 + \cdots + x_n + (2^m - n)A_n}{2^m} \geq (x_1 \cdots x_n)^{1/2^m} A_n^{(2^m-n)/2^m}.$$

The left hand side is just A_n; hence, dividing both sides by $A_n^{(2^m-n)/2^m}$ yields $A_n^{n/2^m} \geq (x_1 \cdots x_n)^{1/2^m}$. Raising both sides to the power of $2^m/n$ yields

$$A_n \geq (x_1 \cdots x_n)^{1/n}.$$

This is what we wanted to show.