Unit 2: The Integral

Pset 2
Due September 24 (4 points each)

(1) page 57: 9de (You may use 9abc as you already proved those for recitation)
(2) page 60: 6
(3) page 70: 7
(4) page 70: 11ac
(5) Prove, using properties of the integral, that for $a, b > 0$

$$\int_1^a \frac{1}{x}dx + \int_1^b \frac{1}{x}dx = \int_1^{ab} \frac{1}{x}dx.$$

Define a function $f(w) = \int_1^w \frac{1}{x}dx$, for $w \in \mathbb{R}^+$. Rewrite the equation above in terms of the function f. Give an example of a function that has the same property as the one displayed here by f.
(6) Suppose we define $\int_a^b s(x)dx = \sum s_k(x_{k-1} - x_k)^2$ for a step function $s(x)$ with partition $P = \{x_0, x_1, \ldots, x_n\}$. Is this integral well-defined? That is, will the value of the integral be independent of the choice of partition? (If well-defined, prove it. If not well-defined, provide a counterexample.)

Bonus:
Define the function (where n is in the positive integers)

$$f(x) = \begin{cases} x & : x = \frac{1}{n^2} \\ 0 & : x \neq \frac{1}{n^2} \end{cases}$$

Prove that f is integrable on $[0, 1]$ and that $\int_0^1 f(x)dx = 0$.