18.014 Problem Set 2 Solutions

Total: 24 points

Problem 1: Let $f(x) = \sum_{k=0}^{n} c_k x^k$ be a polynomial of degree n.
(d) If $f(x) = 0$ for $n + 1$ distinct real values of x, then every coefficient c_k of f is zero and $f(x) = 0$ for all real x.
(e) Let $g(x) = \sum_{k=0}^{m} b_k x^k$ be a polynomial of degree m where $m \geq n$. If $g(x) = f(x)$ for $m + 1$ distinct real values of x, then $m = n$, $b_k = c_k$ for each k, and $g(x) = f(x)$ for all real x.

Solution (4 points)
We prove (d) by induction on n. Since the statement is true for all integers $n \geq 0$, our base case is $n = 0$. If f is a polynomial of degree 0, then $f = c$ is a constant. If f has $n + 1 = 1$ real roots, then $f(x) = 0$ for some x; hence, $c = 0$ and $f(x) = 0$ for all x.
Assume the statement is true for all polynomials of degree n; we prove the statement for a polynomial f of degree $n + 1$. By hypothesis, f has $n + 2$ distinct real roots, $\{a_1, \ldots, a_{n+2}\}$. Using part (c) of this problem (which we did together in recitation),

$$f(x) = (x - a_{n+2}) f_n(x)$$

where f_n is a polynomial of degree n. But, the roots $\{a_1, \ldots, a_{n+2}\}$ are distinct; hence, $a_i - a_{n+2} \neq 0$ if $i < n+2$ and $f_n(a_i) = 0$ for $i = 1, \ldots, n+1$. Thus, by the induction hypothesis, $f_n = 0$ because f_n is a polynomial of degree n with $n+1$ distinct real roots, $\{a_l, \ldots, a_{n+1}\}$. Since $f_n = 0$, we conclude $f(x) = (x - a_{n+2}) f_n(x) = 0$ for every real x. Moreover, since every coefficient of f_n is zero, every coefficient of f is zero. This is what we wanted to show.

(e). If $g(x) = f(x)$ for $m + 1$ distinct values of x, then $g(x) - f(x)$ has $m + 1$ distinct real roots. Moreover, since $\deg f = n$, $\deg g = m$, and $m \geq n$, we observe $\deg(g - f) \leq m$. Thus, by part (d), $g(x) - f(x) = 0$ and every coefficient of $g(x) - f(x)$ is zero. This implies $g(x) = f(x)$ for all real x; moreover, since the coefficients of $g(x) - f(x)$ are $b_k - a_k$, it implies $b_k = a_k$ for all k and $m = \deg g = \deg f = n$.

Problem 2: Let $A = \{1, 2, 3, 4, 5\}$, and let $\mathcal{M}$ denote the set of all subsets of A. For each set S in $\mathcal{M}$, let $n(S)$ defnote the number of elements of S. If $S = \{1, 2, 3, 4\}$

and $T = \{3, 4, 5\}$, compute $n(S \cup T)$, $n(S \cap T)$, $n(S - T)$, and $n(T - S)$. Prove that the set function n satisfies the first three axioms for area.

Solution (4 points)
Note $S \cup T = \{1, 2, 3, 4, 5\}$, $S \cap T = \{3, 4\}$, $S - T = \{1, 2\}$, and $T - S = \{5\}$; hence, $n(S \cup T) = 5$, $n(S \cap T) = 2$, $n(S - T) = 2$, and $n(T - S) = 1$.
For the first axiom, $n(S) \geq 0$ for all sets S since the cardinality of a set is always a positive integer or zero.
For the second axiom, note

$$S \cup T = (S - T) \cup (T - S) \cup (T \cap S)$$

and the union is disjoint. In words, if x is in either S or T, then x is either in S and not in T, in T and not in S, or in S and in T. Further, x can only satisfy one of these conditions at a time. Hence, we count

$$n(S \cup T) = n(S - T) + n(T - S) + n(S \cap T). \qquad (*)$$

Similarly, $T = (T - S) \cup (S \cap T)$ and $S = (S - T) \cup (S \cap T)$ are disjoint unions; thus, $n(T) = n(T - S) + n(S \cap T)$ and $n(S) = n(S - T) + n(S \cap T)$. Solving for $n(S - T)$, $n(T - S)$ and plugging back into our expression $(*)$, we get

$$n(S \cup T) = n(S) + n(T) - n(S \cap T).$$

Third, suppose $S \subset T$ are two subsets of A. Because $S \subset T$, we have $S - T = \emptyset$ and $n(S - T) = 0$. Further, $S \cap T = S$ and $S \cup T = T$. Plugging back into the expression $(*)$, we get $n(T) = 0 + n(T - S) + n(S)$. Solving for $n(T - S)$ yields

$$n(T - S) = n(T) - n(S).$$

Problem 3: (a) Compute $\int_0^9 \lfloor \sqrt{t} \rfloor dt$.
(b) If n is a positive integer, prove $\int_0^{n^2} \lfloor \sqrt{t} \rfloor dt = \frac{n(n-1)(4n+1)}{6}$.

Solution (4 points)
(a). Note

$$\lfloor \sqrt{t} \rfloor = \left\{ \begin{array}{ll} 0 & \text{if } 0 \leq t < 1 \\ 1 & \text{if } 1 \leq t < 4 \\ 2 & \text{if } 4 \leq t < 9 \end{array} \right\}.$$

Thus,

$$\int_0^9 \lfloor\sqrt{t}\rfloor dt = 0\cdot(1-0)+1\cdot(4-1)+2\cdot(9-4) = 13.$$

(b). More generally, $\lfloor\sqrt{t}\rfloor = m$ if $m^2 \le t < (m+1)^2$. Thus,

$$\int_0^{n^2} \lfloor\sqrt{t}\rfloor dt = \sum_{m=0}^{n-1} m\cdot((m+1)^2-m^2).$$

Computing, we get $(m+1)^2-m^2 = 2m+1$; hence, $m\cdot((m+1)^2-m^2) = 2m^2+m$. Using induction on n, we will show

$$\sum_{m=0}^{n-1}(2m^2+m) = \frac{n(n-1)(4n+1)}{6}.$$

This will complete the problem. When $n=1$, both sides are zero. Assume the statement for n; we will prove it for $n+1$. Adding $2n^2+n$ to both sides yields $\sum_{m=0}^{n}(2m^2+m) = \frac{n(n-1)(4n+1)}{6}+2n^2+n$. Computing, the right hand side multiplied by 6 is

$$\begin{aligned} n(n-1)(4n+1)+6(2n^2+n) &= n(4n^2-3n-1+12n+6) \\ &= n(4n^2+9n+5) = n(n+1)((4(n+1)+1). \end{aligned}$$

Thus, we have shown $\sum_{m=0}^{n}(2m^2+m) = \frac{(n+1)n(4(n+1)+1)}{6}$, which is the statement for $n+1$.

Problem 4: If, instead of defining integrals of step functions by using formula (1.3), we used the definition

$$\int_a^b s(x)dx = \sum_{k=1}^{n} s_k^3(x_k-x_{k-1}),$$

a new and different theory of integration would result. Which of the following properties would remain valid in this new theory?
(a) $\int_a^b s + \int_b^c s = \int_a^c s$.
(c) $\int_a^b cs = c\int_a^b s$.

Solution (4 points)
(a) This statement is still true in our new theory of integration; here is a proof. Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of $[a,b]$ such that $s(x)=a_k$ if $x_{k-1}\le x<x_k$. Let

$$b=y_0<y_1<\cdots<y_m=c$$

be a partition of $[b,c]$ such that $s(y)=b_k$ if $y_{k-1}\le y<y_k$. Then by our new definition of the integral of a step function

$$\int_a^b s=\sum_{k=1}^n a_k^3(x_k-x_{k-1}),\ \int_b^c s=\sum_{k=1}^m b_k^3(y_k-y_{k-1}).$$

Now,

$$a=x_0<x_1<\cdots<x_n=b=y_0<\cdots<y_m=c$$

is a partition of $[a,c]$ such that s is constant on each interval. Thus,

$$\int_a^c s=\sum_{k=1}^n a_k^3(x_k-x_{k-1})+\sum_{k=1}^m b_k^3(y_k-y_{k-1}).$$

But, this is just the sum of the integrals $\int_a^b s$ and $\int_b^c s$.

(b) This statement is false for our new theory of integration; here is a counterexample. Suppose $a=0$, $b=1$, s is the constant function 1, and $c=2$. Then

$$\int_0^1 2\cdot 1=2^3(1-0)=8\ne 2=2\cdot(1(1-0))=2\int_0^1 1.$$

Problem 5: Prove, using properties of the integral, that for $a,b>0$

$$\int_1^a \frac{1}{x}dx+\int_1^b \frac{1}{x}dx=\int_1^{ab}\frac{1}{x}dx.$$

Define a function $f(w)=\int_1^w \frac{1}{x}dx$. Rewrite the equation above in terms of f. Give an example of a function that has the same property as f.

Solution (4 points) Using Thm. I.19 on page 81, we have

$$\int_1^b \frac{1}{x}dx=\frac{1}{a}\int_a^{ab}\frac{a}{x}=\int_a^{ab}\frac{1}{x}.$$

And, using Thm. I. 16 on page 81, we have

$$\int_1^a \frac{1}{x}dx+\int_a^{ab}\frac{1}{x}dx=\int_1^{ab}\frac{1}{x}dx.$$

Combining the two gives us the result.
If $f(w) = \int_1^w \frac{1}{x} dx$, then our equation reads $f(a) + f(b) = f(ab)$. The natural logarithm function, $\log(x)$, satisfies this property.

Problem 6: Suppose we define $\int_a^b s(x)dx = \sum s_k(x_{k-1} - x_k)^2$ for a step function $s(x)$ with partition $P = \{x_0, \ldots, x_n\}$. Is this integral well-defined? That is, will the value of the integral be independent of the choice of partition? (If well-defined, prove it. If not well-defined, provide a counterexample.)

Solution (4 points)
This is not a well-defined definition of an integral. Consider the example of the constant function $s = 1$ on the interval $[0, 1]$. If we choose the partition $P = \{0, 1\}$, then we get

$$\int_0^1 1 \cdot dx = 1 \cdot (1-0)^2 = 1.$$

On the other hand, if we choose the partition $P' = \{0, \frac{1}{2}, 1\}$, then we get

$$\int_0^1 1 \cdot dx = 1 \cdot \left(\frac{1}{2} - 0\right)^2 + 1 \cdot \left(1 - \frac{1}{2}\right)^2 = \frac{1}{2}.$$

We get two different answers with two different partitions! Therefore, this integral is not well-defined.

Bonus: Define the function (where n is in the positive integers)

$$f(x) = \left\{ \begin{array}{ll} x & \text{if } x = \frac{1}{n^2} \\ 0 & \text{if } x \neq \frac{1}{n^2} \end{array} \right\}.$$

Prove that f is integrable on $[0, 1]$ and that $\int_0^1 f(x)dx = 0$.

Solution (4 points)
Let $\epsilon > 0$ and choose n such that $n^4 > 1/\epsilon$. Consider a partition of $[0, 1]$ into n subintervals such that $x_0 = 0$ and $x_k = \frac{1}{(n-(k-1))^2}$ for $1 \leq k \leq n$. Define the step function t_n in the following manner: Let $t_n(x_k) = 1$ for all $1 \leq k \leq n$ and $t_n(0) = 0$. For $0 < x < 1/n^2$, let $t_n(x) = 1/n^2$. For all other $x \in [0, 1]$, let $t_n(x) = 0$. Then $t_n(x) \geq f(x)$ for all $x \in [0, 1]$. Moreover, $\int_0^1 t_n(x)dx = 1/n^4 < \epsilon$.

Now, consider $s(x) = 0$ for all $x \in [0, 1]$. Then $s(x) \leq f(x)$ and $\int_0^1 s(x)dx = 0$. By the Riemann condition, f is integrable on $[0, 1]$. Moreover, as $\int_0^1 f(x)dx = \inf\{\int_0^1 t(x)dx | t(x) \geq f(x)$ for step functions $t(x)$ defined on $[0, 1]\}$, we see that $\int_0^1 f(x)dx = 0$.