18.014 Problem Set 3 Solutions

Total: 12 points

Problem 1: Find all values of c for which
(a) $\int_0^c x(1-x)dx = 0$.
(b) $\int_0^c |x(1-x)|dx = 0$.

$\boxed{\text{Solution}}$ (4 points)
(a) Computing, we get

$$\int_0^c x(1-x) = \int_0^c (x-x^2) = \frac{1}{2}c^2 - \frac{1}{3}c^3.$$

Setting the right hand side equal to zero, we get solutions

$$c = \frac{3}{2} \text{ and } c = 0.$$

(b) Observe $x(1-x) \geq 0$ if $0 \leq x \leq 1$ and $x(1-x) \leq 0$ if $x \geq 1$ or $x \leq 0$. There are three cases. If $c < 0$, then

$$\int_0^c |x(1-x)| = \int_0^c -x(1-x) = -\frac{1}{2}c^2 + \frac{1}{3}c^3$$

which is never zero for $c < 0$.
If $0 \leq c \leq 1$, then

$$\int_0^c |x(1-x)| = \int_0^c x(1-x) = \frac{1}{2}c^2 - \frac{1}{3}c^3$$

which is zero only if $c = 0$ in the range $0 \leq c \leq 1$.
Finally, if $c > 1$, then

$$\int_0^c |x(1-x)| = \int_0^1 x(1-x) + \int_1^c -x(1-x).$$

The first integral is $\frac{1}{6}$, and the second integral is non-negative by the comparison theorem (Thm. 1.20) since $-x(1-x) \geq 0$ on the interval $(1, c)$. Hence, this integral is always positive and never zero.
In summary, our integral is only zero when $c = 0$.

Problem 2: Compute the area of the region S between the graphs of $f(x) = x(x^2-1)$ and $g(x) = x$ over the interval $[-1, \sqrt{2}]$. Then make a sketch of the two graphs and indicate S by shading.

$\boxed{\text{Solution}}$ (4 points)
Observe $f(x) = g(x)$ if and only if $x = -\sqrt{2}, 0, \sqrt{2}$. We know this because

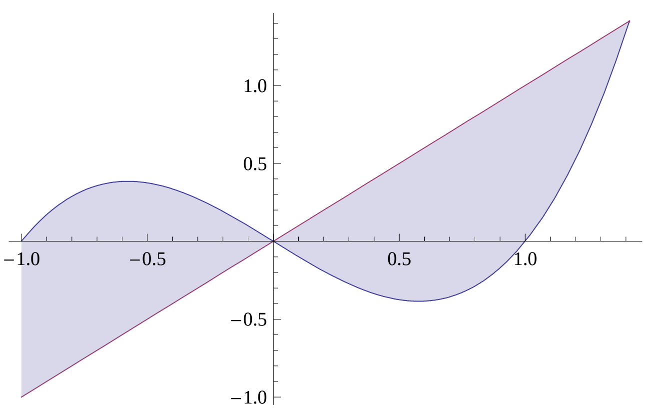
$$f(x) - g(x) = x(x^2-1) - x = x(x+\sqrt{2})(x-\sqrt{2})$$

has roots at $-\sqrt{2}$, 0, and $\sqrt{2}$. Next, observe $f(x) \geq g(x)$ on $[-1, 0]$ and $g(x) \geq f(x)$ on $[0, \sqrt{2}]$. You can check this by testing at a point in each interval and using the intermediate value theorem (which we will learn in chapter three) or by sketching the graph. Then the area between the graphs over the interval $[-1, \sqrt{2}]$ is

$$\int_{-1}^{0} (x(x^2-1) - x)dx + \int_{0}^{\sqrt{2}} (x - x(x^2-1))dx$$

$$= \left(\frac{x^4}{4} - x^2\right)\Big|_{-1}^{0} + \left(x^2 - \frac{x^4}{4}\right)\Big|_{0}^{\sqrt{2}} = \frac{3}{4} + 1 = \frac{7}{4}.$$

Here is a sketch of the graph:



Problem 3: For step functions $s(x)$ and $t(x)$ on $[a, b]$, prove the Cauchy-Schwartz inequality:

$$\left(\int_a^b s(x)t(x)\right)^2 \leq \int_a^b s(x)^2 \int_a^b t(x)^2.$$

Prove that equality holds if and only if $s(x) = ct(x)$ where $c \in \mathbb{R}$.

$\boxed{\text{Solution}}$ (4 points)
By the definition of a step function, there exist partitions $P_1 = \{x_0, \dots, x_n\}$ and $P_2 = \{y_0, \dots, y_m\}$ of $[a, b]$ such that s is constant on the intervals (x_i, x_{i+1}) and t is constant on the intervals (y_j, y_{j+1}). Let $P = P_1 \cup P_2 = \{z_0, \dots, z_l\}$ be the smallest refinement of P_1 and P_2. Then s and t are both constant on the intervals (z_k, z_{k+1}), since each such interval is contained in an interval (x_i, x_{i+1}) and an interval (y_j, y_{j+1}). Define $s(x) = s_k$ and $t(x) = t_k$ for $x \in (z_{k-1}, z_k)$. Then

$$\int_a^b s(x)^2 = \sum_{k=1}^{l} s_k^2(z_k - z_{k-1}), \quad \int_a^b t(x)^2 = \sum_{k=1}^{l} t_k^2(z_k - z_{k-1}), \text{ and}$$

$$\int_a^b s(x)t(x) = \sum_{k=1}^{l} s_k t_k (z_k - z_{k-1}).$$

If we put $a_k = s_k\sqrt{z_k - z_{k-1}}$ and $b_k = t_k\sqrt{z_k - z_{k-1}}$, then the inequality that we are trying to prove becomes

$$\left(\sum_{k=1}^{l} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{l} a_k^2\right)\left(\sum_{k=1}^{l} b_k^2\right).$$

But, this is just the Cauchy-Schwartz inequality on page 42 of your textbook. Moreover, we know that equality holds iff $a_k = cb_k$ for all k and some real number $c \in \mathbb{R}$. Further, $a_k = s_k\sqrt{z_k - z_{k-1}} = ct_k\sqrt{z_k - z_{k-1}} = cb_k$ iff $s_k = ct_k$. Thus, equality holds iff $s = ct$ for some $c \in \mathbb{R}$.

Bonus: Let
$$B = \{x \in [0,1] \big| \ x = \frac{m}{2^n} \text{ some } n, m \in \mathbb{Z}\}.$$
Prove that the function
$$f(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{array} \right\}$$
is not integrable on $[0, 1]$ by our definition of integrability.

$\boxed{\text{Solution}}$ (4 points)
It is enough to prove that for all step functions $s \leq f \leq t$ on $[0, 1]$, we have

$$\int_0^1 (t(x) - s(x))dx \geq 1.$$

Suppose $s \leq f$ is a step function on $[0,1]$ and $s = s_0$ on the interval (x,y). Choose $n \in \mathbb{Z}$ sufficiently large such that $\frac{1}{3^n} < y - x$. Let m be the minimal positive integer such that $m > 3^n x$, and note that $\frac{m}{3^n} \in (x,y)$ since $m < 3^n x + 1 < 3^n y$. But, $f(\frac{m}{3^n}) = 0$; hence, $s_0 \leq 0$. Since this is true for every interval on which s is constant, we must have $s \leq 0$ on $[0,1]$. By the comparison theorem (Thm. 1.5) and the homogeneous property (Thm. 1.3), we conclude

$$\int_0^1 -s(x)dx \geq 0$$

for any step function $s \leq f$ on $[0,1]$.
Now, suppose $f \leq t$ is a step function on $[0,1]$ and $t = t_0$ on the interval (x,y). Choose $n \in \mathbb{Z}$ sufficiently large such that $\frac{1}{2^n} < y - x$. Then we can find $m \in \mathbb{Z}$ such that $\frac{m}{2^n} \in (x,y)$. Since $f(\frac{m}{2^n}) = 1$, we must have $t_0 \geq 1$. Thus, $t \geq 1$ and

$$\int_0^1 t(x)dx \geq 1$$

by the comparison theorem (Thm. 1.5). Putting these together, we get

$$\int_0^1 (t(x) - s(x))dx = \int_0^1 t(x)dx + \int_0^1 -s(x)dx \geq 1 + 0 = 1.$$