18.014 Problem Set 4 Solutions

Total: 24 points

Problem 1: Establish the following limit formulas. You may assume the formula $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$.
(a)

$$\lim_{x\to 0}\frac{\sin(5x)}{\sin(x)} = 5.$$

(b)

$$\lim_{x\to 0}\frac{\sin(5x)-\sin(3x)}{x} = 2.$$

(c)

$$\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x^2} = \frac{1}{2}.$$

Solution (4 points)
(a) Using the product formula for limits (Thm. 3.1 part iii), we have

$$\lim_{x\to 0}\frac{\sin(5x)}{\sin(x)} = \lim_{x\to 0}\frac{\sin(5x)}{5x}\frac{5x}{\sin(x)} = \lim_{x\to 0}\frac{\sin(5x)}{5x}\cdot\lim_{x\to 0}\frac{5x}{\sin(x)} = AB.$$

For the first term, note that $5x$ approaches zero as x approaches zero; hence, $A = 1$ by the assumed limit formula. For the second term, note $\lim_{x\to 0}\frac{5x}{\sin(x)} = 5\lim_{x\to 0}\frac{1}{\sin(x)/x} = 5\cdot 1 = 5$ by the product rule and the quotient rule (Thm. 3.1 part iv). Thus, $B = 5$ and

$$\lim_{x\to 0}\frac{\sin(5x)}{\sin(x)} = 5$$

as desired.

(b) Here we use the difference rule (Thm. 3.1 part ii) to obtain

$$\lim_{x\to 0}\frac{\sin(5x)-\sin(3x)}{x} = \lim_{x\to 0}\frac{\sin(5x)}{x} - \lim_{x\to 0}\frac{\sin(3x)}{x}.$$

Next, for any real number $a \neq 0$, we observe

$$\lim_{x\to 0}\frac{\sin(ax)}{x} = a\lim_{x\to 0}\frac{\sin(ax)}{ax} = a\lim_{x\to 0}\frac{\sin(x)}{x} = a.$$

Plugging back into the above formula yields

$$\lim_{x\to 0} \frac{\sin(5x) - \sin(3x)}{x} = \lim_{x\to 0} \frac{\sin(5x)}{x} - \lim_{x\to 0} \frac{\sin(3x)}{x} = 5 - 3 = 2.$$

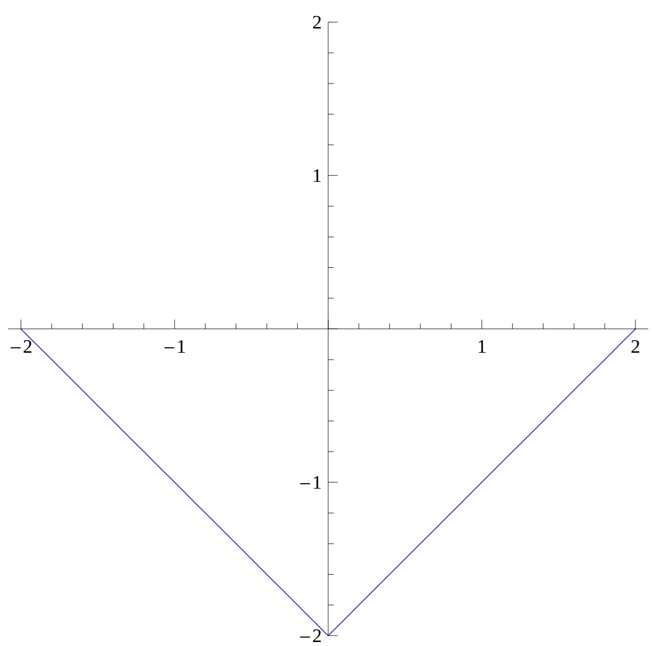
(c) We use the product rule to get

$$\lim_{x\to 0} \frac{1 - \sqrt{1-x^2}}{x^2} = \lim_{x\to 0} \frac{(1 - \sqrt{1-x^2})(1 + \sqrt{1-x^2})}{x^2} \lim_{x\to 0} \frac{1}{(1 + \sqrt{1-x^2})}.$$

Since $(1 - \sqrt{1-x^2})(1 + \sqrt{1-x^2}) = 1 - (1 - x^2) = x^2$, the first limit is one. For the second limit, note that $\frac{1}{1+\sqrt{1-x^2}}$ is the composition of the functions $1 - x^2$, $\sqrt{x}$, $1 + x$, and $\frac{1}{x}$, which are continuous at the points 0, 1, 1, and 2 by Example 5 and Theorem 3.2. Hence, by Theorem 3.5, the function $\frac{1}{1+\sqrt{1-x^2}}$ is continuous at $x = 0$, and we can just plug in $x = 0$ to get that the limit of the second term is $1/2$. Multiplying the two terms together, the limit is $1 \cdot 1/2 = 1/2$.

Problem 2: Let $A(x) = \int_{-2}^{x} f(t)dt$ where $f(t) = -1$ when $t < 0$ and $f(t) = 1$ when $t \geq 0$. Graph $y = A(x)$ when $x \in [-2, 2]$. Using ϵ and δ, show that $\lim_{x\to 0} A(x)$ exists and find its value.

Solution (4 points)
Here is the graph:



Now, we prove $\lim_{x\to 0} A(x) = -2$.
Given $\epsilon > 0$, let $\delta = \epsilon$. Suppose $|x| = |x - 0| < \delta$, and observe that there are two possibilities, $x \le 0$ or $x \ge 0$. In the first case, we have $A(x) = \int_{-2}^{x}(-1)dx = -x - 2$. Hence,

$$|A(x) - (-2)| = |(-x - 2) + 2| = |-x| = |x| < \delta = \epsilon.$$

In the second case, $A(x) = \int_{-2}^{0}(-1)dx + \int_{0}^{x} 1dx = -2 + x$. Hence,

$$|A(x) - (-2)| = |(-2 + x) + 2| = |x| < \delta = \epsilon.$$

In particular, this means $A(x)$ is continuous (we should have known this already because of Thm. 3.4).

Problem 3: Let $f(x)$ be defined for all x, and continuous except for $x = -1$ and $x = 3$. Let

$$g(x) = \left\{ \begin{array}{c} x^2 + 1 \text{ for } x > 0 \\ x - 3 \text{ for } x \le 0 \end{array} \right\}.$$

For what values of x can you be sure that $f(g(x))$ is continuous? Explain.

Solution (4 points) We wish to use Theorem 3.5 to show that $f(g(x))$ is continuous for some values x. But, we can only use the theorem when g is continuous at x and f is continuous at $g(x)$. Since g is piecewise polynomial, we know g is continuous except at $x = 0$ by example one. Now, g takes the value 3 at $x = \sqrt{2}$, and it never takes the value -1. Hence, at all values except possibly $x = 0$ and $x = \sqrt{2}$, we know that $f(g(x))$ is continuous.

This isn't part of the solution, but for the record you might want to know that $f(g(x))$ will be continuous at $x = 0$ if and only if $f(1) = f(-3)$. On the other hand, $f(g(x))$ can never be continuous at $x = \sqrt{2}$. It's a good exercise to prove these statements using ϵ-δ arguments.

Problem 4: Suppose that g, h are continuous functions on $[a, b]$. Suppose there exists $c \in (a, b)$ such that $g(c) = h(c)$. Define a function $f(x)$ on $[a, b]$ such that $f(x) = g(x)$ for $x < c$ and $f(x) = h(x)$ for $x \ge c$. Prove that $f(x)$ is continuous on $[a, b]$.

Solution (4 points) We divide the task of showing that $f(x)$ is continuous in $[a, b]$ into three cases. First, if $a \le x_0 < c$, then $f(x_0) = g(x_0)$. Given $\epsilon > 0$, we can find $\delta_1 > 0$ such that $|x - x_0| < \delta_1$ implies $|g(x) - g(x_0)| < \epsilon$ since g is continuous at x_0. If we put $\delta = \min\{\delta_1, x_0 - a, c - x_0\}$, then $|x - x_0| < \delta$ implies that $f(x) = g(x)$ and

$|g(x) - g(x_0)| < \epsilon$. Thus, we conclude $|f(x) - f(x_0)| < \epsilon$ and f is continuous at x_0.

Next, we consider the case $c < x_0 < b$, and we note that $f(x_0) = h(x_0)$. Given $\epsilon > 0$, we can find $\delta_1 > 0$ such that $|x - x_0| < \delta_1$ implies $|h(x) - h(x_0)| < \epsilon$. If we define $\delta = \min\{\delta_1, b - x_0, x_0 - c\}$, then $|x - x_0| < \delta$ implies $f(x) = h(x)$ and $|h(x) - h(x_0)| < \epsilon$. Thus, $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$ and f is continuous at $x = x_0$.

Finally, we consider the case of continuity at c. Given $\epsilon > 0$, there exists δ_1 such that $|x - c| < \delta_1$ implies $|g(x) - g(c)| < \epsilon$ because g is continuous at c. There also exists δ_2 such that $|x - c| < \delta_2$ implies $|h(x) - h(c)| < \epsilon$ because h is continuous at c. Put $\delta = \min\{\delta_1, \delta_2\}$. If $|x - c| < \delta$, then there are two possibilites. If $x \leq c$, then $f(x) = g(x)$ and $|x - c| < \delta$ implies $|x - c| < \delta_1$ and $|g(x) - g(c)| < \epsilon$. Thus, $|f(x) - f(c)| < \epsilon$. If $x > c$, then $f(x) = h(x)$ and $|x - c| < \delta$ implies $|x - c| < \delta_2$ and $|h(x) - h(c)| < \epsilon$. Thus, $|f(x) - f(c)| < \epsilon$ using $f(c) = g(c) = h(c)$. Regardless of case, we realize $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$. Thus, f is continuous at c.

Problem 5: Let $f(x) = \sin(1/x)$ for $x \in \mathbb{R}$, $x \neq 0$. Show that for any $a \in \mathbb{R}$, the function $g(x)$ defined by

$$g(x) = \left\{ \begin{array}{cc} f(x) & \text{for } x \neq 0 \\ a & \text{for } x = 0 \end{array} \right\}$$

is not continuous at $x = 0$.

Solution (4 points) Observe that if $x = \frac{1}{\pi n/2}$ where $n = 4k + 1$ with $k \in \mathbb{P}$, then

$$\sin(1/x) = \sin(\pi n/2) = \sin((4k + 1)\pi/2) = \sin(\pi/2) = 1.$$

On the other hand, if $x = \frac{1}{\pi n/2}$ where $n = 4k + 3$ with $k \in \mathbb{P}$, then

$$\sin(1/x) = \sin(\pi n/2) = \sin((4k + 3)\pi/2) = \sin(3\pi/2) = -1.$$

Now, suppose $a \neq 1$ and $g(x)$ is continuous at $x = 0$. Take $\epsilon = |1-a|$. Then there must exist δ such that $|x - 0| < \delta$ implies $|g(x) - a| < \epsilon$. But, by the archimedean property of the real numbers, we can always choose $x = \frac{1}{\pi n/2} < \delta$ with $n = 4k + 1$ and $k \in \mathbb{P}$. Thus, we must have $|g(x) - a| < \epsilon$. But, $g(x) = 1$ and we assumed $\epsilon = |1 - a| = |g(x) - a|$. This is a contradiction, and we conclude that $g(x)$ is not continuous at $x = 0$.

We handle the case $a = 1$ similarly. Still assuming $g(x)$ is continuous at $x = 0$, we take $\epsilon = 2$. Then there must exist δ such that $|x| < \delta$ implies $|g(x) - 1| < \epsilon$.

But, choosing $x = \frac{1}{(4k+3)\pi/2} < \delta$, we note $g(x) = -1$ and $|g(x) - 1| = 2$, which isn't less than $\epsilon = 2$. Thus, $g(x)$ is still not continuous at $x = 0$.

Problem 6: Let f be a real-valued function, which is continuous on the closed interval $[0, 1]$. Assume that $0 \leq f(x) \leq 1$ for each $x \in [0, 1]$. Prove that there is at least one point c in $[0, 1]$ for which $f(c) = c$. Such a point is called a fixed point of f.

Solution (4 points) Put $g(x) = f(x) - x$. Then $g(0) = f(0) \geq 0$, and $g(1) = f(1) - 1 \leq 0$ since $f(1) \leq 1$. Hence, $g(0)$ and $g(1)$ have opposite signs and we may apply Bolzano's Theorem (Thm. 3.6). Therefore, there exists $c \in [0, 1]$ such that $f(c) - c = g(c) = 0$. Moving c to the other side of the equation tell us that $f(c) = c$ as desired.

Bonus: Let f be a bounded function that is integrable on $[a, b]$. Prove that there exists $c \in [a, b]$ such that

$$\int_a^b f(x)dx = 2\int_a^c f(x)dx.$$

Solution (4 points) Define $g(t) = 2\int_a^t f(x)dx - \int_a^b f(x)dx$. By Theorem 3.4 and Theorem 3.2, $g(t)$ is a continuous function on $[a, b]$. Note that $g(a) = -\int_a^b f(x)dx$ and $g(b) = 2\int_a^b f(x)dx - \int_a^b f(x)dx = \int_a^b f(x)dx$. Hence, $g(b) = -g(a)$ and g takes values with opposite signs at a and b. (Or $g(b) = g(a) = 0$. In that case just choose $a = c$.) Thus, we may apply Bolzano's theorem, and we find that there exists $c \in [a, b]$ such that

$$2\int_a^c f(x)dx - \int_a^b f(x)dx = g(c) = 0.$$

For this value of c, we have the desired formula $\int_a^b f(x)dx = 2\int_a^c f(x)dx$.