18.014 Problem Set 5 Solutions

Total: 24 points

Problem 1: Let $f(x) = x^4 + 2x^2 + 1$ for $0 \le x \le 10$.
(a) Show f is strictly increasing; what is the domain of its inverse function g?
(b) Find an expression for g, using radicals.

Solution (4 points) (a) Let $0 \le x < y \le 10$. Since

$$y^4 - x^4 = (y - x)(y^3 + y^2x + yx^2 + x^3) > 0,$$

we have $y^4 > x^4$. Similarly, $y^2 > x^2$ since

$$y^2 - x^2 = (y - x)(y + x) > 0.$$

Summing, we get

$$f(y) = y^4 + 2y^2 + 1 > x^4 + 2x^2 + 1$$

and f is strictly increasing. Since $f(0) = 1$, $f(10) = 10,201$, and f is strictly increasing, the domain of its inverse function g is

$$\{x \mid 1 \le x \le 10,201\}.$$

(b) Observe $g(x) = \sqrt{\sqrt{x} - 1}$. It's a good exercise to check that $f(g(x)) = g(f(x)) = x$.

Problem 2: (a) Show by example that the conclusion of the extreme value theorem can fail if f is only continuous on $[a, b)$ and bounded on $[a, b]$.
(b) Let $f(x) = x$ for $0 \le x < 1$; let $f(1) = 5$. Show that the conclusion of the small span theorem fails for the function $f(x)$.

Solution (4 points) (a) Define $f(x) = x$ if $0 \le x < 1$, and define $f(1) = 0$. Let $x \in [0, 1]$. We claim $f(x)$ is not a maximum value of the function f on $[0, 1]$. If $x = 1$, then $f(1) = 0$, and 0 is not a maximum of f on $[0, 1]$ since $f(\frac{1}{2}) = \frac{1}{2} > 0$. If $x \ne 1$, then define

$$y = \frac{1 + x}{2}.$$

Note that $f(x) = x < y = f(y)$; hence, $f(x)$ is not a maximum of f on $[0, 1]$. We conclude that f has no maximum on $[0, 1]$.

(b) Suppose that the conclusion of the small span theorem is true for the function $f(x)$ in part (b). Then given $\epsilon = 1$, we can find a partition $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ of the interval $[0, 1]$ such that whenever $x_{i-1} \leq x < y \leq x_i$, we have

$$|f(x) - f(y)| < 1.$$

Consider the interval $[x_{n-1}, x_n] = [x_{n-1}, 1]$, and put $y = 1$. For any $x_{n-1} \leq x < 1$, we have

$$|f(x) - f(y)| = |x - 5| = 5 - x > 4$$

since $x < 1 < 5$. This is a contradiction. Thus, the conclusion of the small span theorem fails for this function.

Problem 3: Assume f is continuous on $[a, b]$. Assume also that $\int_a^b f(x)g(x)dx = 0$ for every function g that is continuous on $[a, b]$. Prove that $f(x) = 0$ for all x in $[a, b]$.

Solution (4 points) Put $g(x) = f(x)$. Then $\int_a^b f(x)^2 dx = 0$. Since f is a continuous function, x^2 is a continuous function, and the composition of continuous functions is continuous, we know $f(x)^2$ is a continuous function. Since $f(x)^2$ is continuous on the interval $[a, b]$, we know $f(x)^2$ is bounded and integrable on the interval $[a, b]$ by Theorem 3.11 and Theorem 3.14. Thus, we may apply problem 7 on page 155, and we get that $f(x)^2 = 0$ for every $x \in [a, b]$. It follows that $f(x) = 0$ for all $x \in [a, b]$.

Problem 4: We define a set $A \subset \mathbb{R}$ to be dense in $\mathbb{R}$ if every open interval of $\mathbb{R}$ contains at least one element of A. Let A be a dense subset of $\mathbb{R}$, and let $f(x)$ be a continuous function such that $f(x) = 0$ for all $x \in A$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

Solution (4 points) Fix $x \in \mathbb{R}$. To show $f(x) = 0$, it is enough to show $|f(x)| < \epsilon$ for any $\epsilon > 0$. So fix $\epsilon > 0$. Since f is continuous at x, there exists δ such that $y \in (x - \delta, x + \delta)$ implies $|f(y) - f(x)| < \epsilon$. But, the interval $(x - \delta, x + \delta)$ must contain $y \in A$. For this y, we have $f(y) = 0$. Hence, $|0 - f(x)| < \epsilon$ and $|f(x)| < \epsilon$ as desired.

Problem 5: Let $f(x)$ be a continuous function on $[0, 1]$ and fix $w \in \mathbb{R}$. Show that there exists $z \in [0, 1]$ such that the distance between $(w, 0)$ and the curve $y = f(x)$ is minimized by $(z, f(z))$.

Solution (4 points) Note that the distance between $(x, f(x))$ and $(w, 0)$ is $g(x) = \sqrt{(x-w)^2 + f(x)^2}$ by the Pythagorean theorem. Observe that $(x-w)^2$ is a continuous function because it is a polynomial in x, $f(x)^2$ is a continuous function because it is the composition of continuous functions, $(x-w)^2 + f(x)^2$ is continuous because it is the sum of continuous functions, and finally $g(x) = \sqrt{(x-w)^2 + f(x)^2}$ is continuous because it is the composition of continuous functions. Because $g(x)$ is a continuous function on $[0, 1]$ it must have a minimum value $z \in [0, 1]$ by the extreme value theorem. Hence, $(z, f(z))$ minimizes the distance between the curve $y = f(x)$ and the point $(w, 0)$.

Problem 6: Show that the line $y = -x$ is tangent to the curve given by the equation $y = x^3 - 6x^2 + 8x$. Find the point of tangency. Does this tangent line intersect the curve anywhere else?

Solution (4 points) First, we figure out where the curves $y = -x$ and $y = x^3 - 6x^2 + 8x$ intersect. Setting them equal yields $-x = x^3 - 6x^2 + 8x$. Rearanging and factoring, we get

$$x(x-3)^2 = 0.$$

Thus, the two curves intersect at $x = 3$ and $x = 0$.
Next, we determine where the two curves have the same derivative. In the case $y = -x$, we get $\frac{dy}{dx} = -1$. In the case of $y = x^3 - 6x^2 + 8x$, we get $\frac{dy}{dx} = 3x^2 - 12x + 8$. Setting these two equal yields

$$0 = 3x^2 - 12x + 9 = 3(x-3)(x-1).$$

Thus, the curves share the same slope at $x = 1$ and $x = 3$. The curve $y = -x$ is tangent to the curve $y = x^3 - 6x^2 + 8x$ when the two share the same value and derivative. This happens only at the point $x = 3$. The curves also intersect at the point $x = 0$, but they do not share the same slope at that point.

Bonus: Define a function f on the interval $[0, 1]$ by setting $f(x) = 0$ if x is irrational, $f(x) = \frac{1}{n}$ if $x = \frac{m}{n}$ with m and n positive integers having no common factors except one, and $f(0) = 1$.
(a) Show that f is integrable on $[0, 1]$.
(b) Show that f is continuous at every irrational and discontinuous at every rational.

Solution (4 points) (a) To show that f is integrable on $[0, 1]$, we must show that its upper integral $\overline{I}(f)$ and its lower integral $\underline{I}(f)$ agree. We know $\overline{I}(f) \geq \underline{I}(f)$; hence, we must show the opposite inequality.

First, observe that 0 is a step function and $0 \le f$ on $[0,1]$. Thus, $0 \le \underline{I}(f)$. Now, we bound $\overline{I}(f)$ from above. We introduce step functions s_n for $n = 2, 3, \ldots$ as follows.

Fix $n \in \mathbb{P}$, and let

$$P = \left\{ \frac{p}{q} \pm \frac{1}{n^3} \in [0,1] \middle| \; q < n, \; p, q \in \mathbb{P} \right\}.$$

Since the set P is finite, it yields a partition of $[0,1]$. Define $s_n(x) = 1$ if there exist $p, q \in \mathbb{P}$ with $q < n$ such that $\frac{p}{q} - \frac{1}{n^3} < x < \frac{p}{q} + \frac{q}{n^3}$. Let $s_n(x) = 0$ if there do not exist such p and q. Clearly s_n is a step function with respect to the partition P.

Observe that for fixed $q < n$, the number of $\frac{p}{q} \in [0,1]$ is at most $q+1$, which is at most n. Moreover, there are less than n positive integers q such that $q < n$. Thus, there exist no more than n^2 intervals $(\frac{p}{q} - \frac{1}{n^3}, \frac{p}{q} + \frac{1}{n^3})$ in the interval $[0,1]$. Thus, we may bound

$$\int_0^1 s_n(x) \le n^2 \frac{1}{n^3} = \frac{1}{n}.$$

But, then $\overline{I}(f) \le \frac{1}{n}$ for all $n \in \mathbb{P}$. By the archimedean property of the reals, this implies that $\overline{I}(f) \le 0$. Then

$$0 \le \underline{I}(f) \le \overline{I}(f) \le 0$$

implies $\underline{I}(f) = \overline{I}(f) = 0$ and f is integrable on $[0,1]$.

(b) Let $\alpha \in [0,1]$ be an irrational number. Given $\epsilon > 0$, choose $n \in \mathbb{P}$ such that $\frac{1}{n} < \epsilon$. As remarked in part (a), there are finitely many rational numbers $\frac{p}{q} \in [0,1]$ such that $q < n$. Let δ be the minimum of the distances between $\frac{p}{q}$ and α for $q < n$. Since α is irrational, none of these distances are zero; hence, $\delta > 0$.

If $|x - \alpha| < \delta$, then there are two options for $f(x)$. If x is irrational, then $f(x) = 0$. If x is rational, then $x = \frac{p}{q}$ with p and q having no common factors except one and $q \ge n$, since $|x - \alpha| < \delta$. Thus, $f(x) \le \frac{1}{n} < \epsilon$. Either way, we get

$$|f(x) - f(\alpha)| = |f(x)| < \epsilon.$$

We have shown that f is continuous at α.

Next, let $x = \frac{m}{n}$ be a rational number in lowest terms (m and n have no common factors except one). Put $\epsilon = \frac{1}{2n}$. Assume f is continuous at x. Then there exists $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$. As remarked above, there are finitely many $\frac{p}{q} \in [0,1]$ with $q \le n$ and $p, q \in \mathbb{P}$. Hence, there exists a minimal distance d such that whenever $|y-x|<d$ and $y \neq x$, we cannot have $y=\frac{p}{q}$ with $q<n$. Let $d_1=\frac{1}{2}\min\{d,\delta\}$. Then $y=x+d_1$ satisfies $|y-x|<\delta$. There are two possibilities for $f(y)$. If y is irrational, then $f(y)=0$. If $y=\frac{p}{q}$ is rational and in lowest terms, then $q>2n$. Hence, $f(y)<\frac{1}{2n}$. So regardless of case,

$$|f(y)-f(x)|=|f(y)-\frac{1}{n}|>\frac{1}{2n}=\epsilon.$$

This contradicts our assumption. We conclude that f is not continuous at any rational number.