# 18.014 Problem Set 6 Solutions

Total: 24 points

**Problem 1:** A water tank has the shape of a right-circular cone with its vertex down. Its alititude is 10 feet and the radius of the base is 15 feet. Water leaks out of the bottom at a constant rate of 1 cubic foot per second. Water is poured into the tank at a constant rate of $c$ cubic feet per second. Compute $c$ so that the water level will be rising at a rate of 4 feet per second at the instant when the water is 2 feet deep.

Solution (4 points) The relationship of interest in this problem is $V = \pi r^2 h/3$, where $V, r, h$ are, respectively, volume of water in the cone, radius of the cone at the maximum water height, and height of the water in the cone. Notice that all known quantities are related to volume and height, so determining $V$ in terms of $h$ will make things a bit simpler. Using similar triangles, we see that $h/10 = r/15$ or $3h/2 = r$. Substituting we get, $V = 3\pi h^3/4$. Now, as both $h$ and $V$ depend on $t$ we compute

$$\frac{dV}{dt} = \frac{9\pi h^2}{4}\frac{dh}{dt}.$$

Now, $h = 2ft$, $dh/dt = 4ft/s$, and $dV/dt = (c-1)ft^3/s$. That is,

$$c - 1 = \frac{9\pi \cdot 4}{4} \cdot 4 = 36\pi.$$

Thus we need $c = (36\pi + 1)ft^3/s$.

**Problem 2:** Let $f(x) = 1 - x^{2/3}$. Show that $f(1) = f(-1) = 0$ but that $f'(x)$ is never zero in the interval $[-1, 1]$. Explain how this is possible in view of Rolle's theorem.

Solution (4 points) As $(-1)^{2/3} = ((-1)^{1/3})^2 = 1$, the equalities $f(1) = f(-1) = 0$ follow trivially. From Example 6 (page 163), we can set $n = 3$ and consider the limit for all $x \neq 0$. (Recall we have defined the n-th root for $x < 0$ when $n$ is odd.) Then, following the argument of Example 3 (page 166), we see $f'(x) = -2/3x^{-1/3}$ for all $x \neq 0$. In particular, wherever $f'$ is defined, $f' \neq 0$.

Rolle's theorem states that for a continuous function $f$ on $[a, b]$ that is differentiable on $(a, b)$, if $f(a) = f(b)$ then there exists $c \in (a, b)$ such that $f'(c) = 0$. The reason that the conclusion of Rolle's theorem fails here is that one of the hypothesis of Rolle's theorem is not true for the function $f$. Namely, $f$ is not differentiable on all of $(-1,1)$, as $f$ is not differentiable at zero.

**Problem 3:** Let $f(x) = \frac{x}{1+x^2}$.
(a) Find all points x such that $f'(x) = 0$.
(b) Examine the sign of $f'$ and determine those intervals in which $f$ is monotonic.
(c) Examine the sign of $f''$ and determine those intervals in which $f'$ is monotonic.
(d) Make a sketch of the graph of $f$.

Solution (4 points) (a) Notice first that $g(x) = x$ and $h(x) = 1+x^2$ are differentiable for all $x$ by Example 3 (page 161). Moreover, as $a^2 > 0$ for all $a \neq 0$, $h(x) > 0$ for all $x$. So, we can freely apply Theorem 4.1 (the quotient theorem for derivatives) to see

$$f'(x) = \frac{1+x^2 - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}.$$

Thus, $f'(x) = 0$ at $x = \pm 1$.
(b) The value of $1-x^2$ determines the sign of $f'(x)$. First, we proved earlier that polynomials are continuous functions. By the intermediate value theorem, if $f(x_1) > 0$ and $f(x_2) < 0$, there exists some $x$ in the interval with endpoints $x_1, x_2$ such that $f(x) = 0$. By work in (a), we see that $f'(x)$ must have a fixed sign on each of the intervals $(-\infty,-1), (-1,1), (1,\infty)$. Observe $1 - x^2 > 0$ implies $1^2 = 1 > x^2 > 0$. Since the square root function is strictly increasing, it follows that $1 - x^2 > 0$ only if $1 > |x|$. That is, $f'(x) > 0$ on $(-1,1)$ and $f'(x) < 0$ on $(-\infty,-1) \cup (1,\infty)$. By Theorem 4.7, $f$ is strictly decreasing on $(-\infty,-1]$, strictly increasing on $[-1,1]$, and strictly decreasing again on $[1,\infty)$.
(c) We compute the second derivative again using the quotient theorem. As before, since the function in the denominator is always positive, we have no trouble computing

$$f''(x) = \frac{-2x(1+x^2)^2 - (1-x^2)\cdot 2(1+x^2)\cdot 2x}{(1+x^2)^4} = \frac{2x(x^2-3)}{(1+x^2)^3}.$$

Again, the denominator of $f''$ is always positive, so we consider the sign of $2x(x^2-3)$. As before, the continuity of this function and the Intermediate Value Theorem imply we need to consider the intervals $(-\infty,-\sqrt{3})$, $(-\sqrt{3},0)$, $(0,\sqrt{3})$, $(\sqrt{3},\infty)$ since $f''(x) = 0$ for $x = \pm\sqrt{3}, 0$. Observe that $x^2 - 3 < 0$ if $|x| < \sqrt{3}$. Thus, we quickly determine $f'' > 0$ on the intervals $(-\sqrt{3},0)$ and $(\sqrt{3},\infty)$ and negative on the other two intervals. By Theorem 4.7, we see $f'$ is strictly increasing on $[-\sqrt{3},0]$ and $[\sqrt{3},\infty)$ and strictly decreasing on $(-\infty,-\sqrt{3}]$ and $[0,\sqrt{3}]$.

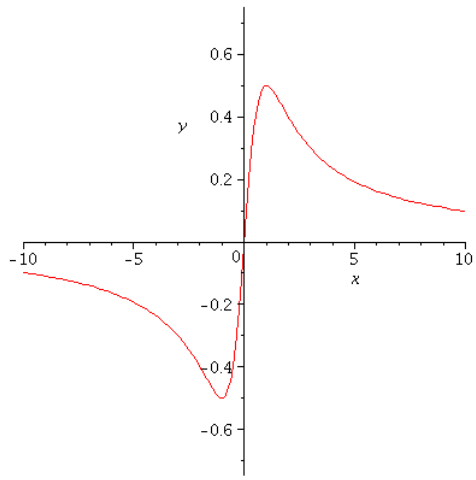


Figure 1: Here's a "sketch" of the graph.

**Problem 4:** Suppose that $f$ is differentiable at $x = c$. Show that $|f|$ is differentiable at $x = c$ provided $f(c) \neq 0$. Give a counterexample when $f(c) = 0$.

Solution (4 points) We begin by observing that if $f$ is differentiable at $x = c$ then by Example 7 (page 163), $f$ is continuous at $x = c$. Also, as $f(c) \neq 0$, Theorem 3.7 implies there exists an interval $I := (c - \delta, c + \delta)$ such that $f(x)$ has the same sign as $f(c)$ for all $x \in I$. First, assume $f(c) > 0$. Then for $|h| < \delta$, $f(x + h) > 0$. Thus,

$$\lim_{h \to 0} \frac{|f|(c + h) - |f|(c)}{h} = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} = f'(c).$$

Now, if $f(c) < 0$ then $|f|(c) = -f(c)$ and for $|h| < \delta$ one has

$$\lim_{h \to 0} \frac{|f|(c + h) - |f|(c)}{h} = \lim_{h \to 0} \frac{-f(c + h) - (-f(c))}{h} = -f'(c).$$

In either case, the limit exists and thus $|f|$ is differentiable at $x = c$.

For the second part of the problem, consider the function $f(x) = x$ at $x = 0$. It is differentiable at $x = 0$ but $|f|(x) = |x|$ is not differentiable at $x = 0$.

**Problem 5:** Let $f(x) = xg(x)$ where $g$ is a continuous function defined on $[-1, 1]$. Prove that $f$ is differentiable at $x = 0$ and find $f'(0)$ in terms of $g$. (The hardest part of this problem will be writing all of the details very carefully. Justify your equalities.)

Solution (4 points) We use the definition of the derivative. That is, we show

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

exists. By definition, this limit is $f'(0)$. First, observe that $f(0) = 0 \cdot g(0) = 0$ and $f(0+h) = h \cdot g(h)$. Thus, $f(h) - f(0) = h \cdot g(h)$. So the difference quotient simplifies to $\frac{h \cdot g(h)}{h} = g(h)$. As $g$ is continuous on $[-1, 1]$, it is certainly continuous at $x = 0$ and thus $\lim_{h \to 0} g(h) = g(0)$. It follows that

$$lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = g(0)$$

and thus $f'(0) = g(0)$.

**Problem 6:** A function $f$ is defined for all real $x$ by the formula

$$f(x) = 3 + \int_0^x \frac{1 + \sin t}{2 + t^2} dt.$$

Without attempting to evaluate this integral, find a quadratic polynomial $p(x) = a + bx + cx^2$ such that $p(0) = f(0)$, $p'(0) = f'(0)$, $p''(0) = f''(0)$.

Solution (4 points) We begin by observing that

$$f(0) = 3 + \int_0^0 \frac{1 + \sin t}{2 + t^2} dt = 3 + 0 = 3.$$

Now, we use the fundamental theorem of calculus. Note that $g(t) = 1 + \sin t$ and $h(t) = 2 + t^2$ are continuous functions. Moreover, $h(t) \neq 0$ for all $t \in \mathbb{R}$. By Theorem 3.2, $g(t)/h(t)$ is continuous for all $t$ and thus we can use the fundamental theorem of calculus to compute $f'(x)$. That is,

$$f'(x) = \frac{d}{dx}(3) + \frac{d}{dx} \int_0^x \frac{1 + \sin t}{2 + t^2} dt = \frac{1 + \sin x}{2 + x^2}.$$

Thus, simply by evaluating the function, $f'(0) = 1/2$. Finally, we find $f''(x)$ by using the quotient theorem for derivatives. As $2 + x^2 > 0$ for all $x$, the theorem applies for all values of $x$ (in particular for $x = 0$). Therefore

$$f''(0) = \frac{\cos x(2 + x^2) - (1 + \sin x)(2x)}{(2 + x^2)^2}\Big|_{x=0} = \frac{2 - 0}{4} = 1/2.$$

Thus, we seek a quadratic polynomial $p(x)$ such that $p(0) = 3, p'(0) = 1/2, p''(0) = 1/2$. Observe that for $p(x) = a + bx + cx^2$, $p(0) = a$, $p'(0) = b$, and $p''(0) = 2c$. Thus, $a = 3,\ b = 1/2,\ c = 1/4$. The polynomial we seek is thus

$$p(x) = 3 + 1/2x + 1/4x^2.$$

**Bonus:** Prove a pseudo-converse to (4). In particular, prove that if $|f|$ is differentiable at $x = c$ and $f$ is continuous at $x = c$, then $f$ is differentiable at $x = c$.

Solution (4 points) We give the solution, with slightly less detail than the previous solutions.

First, observe that if $f(c) \neq 0$ then the continuity of $f$ at $x = c$ implies there exists $\delta > 0$ such that $f(c + h)$ and $f(c)$ have the same sign for all $|h| < \delta$. (This is by Theorem 3.7 as in Problem 4.) In that case, $f'(c) = \pm|f|'(c)$ from simply considering the limit definition depending on the sign of $f(c)$.

Now, suppose $f(c) = 0$. Then

$$\frac{f(c + h) - f(c)}{h} = \frac{f(c + h)}{h}.$$

As $|f|'(c)$ exists,

$$\lim_{h\to 0^-} \frac{|f(c + h)|}{h} = \lim_{h\to 0^+} \frac{|f(c + h)|}{h} = L.$$

Observe that the first limit must be non-positive, as the numerator is non-negative and the denominator is negative. So, $L \leq 0$. The second limit has non-negative numerator and positive denominator so, $L \geq 0$. It follows that $L = 0$. Thus,

$$\lim_{h\to 0} \left|\frac{f(c + h)}{h}\right| = 0.$$

We now appeal to the squeeze theorem. Namely, as

$$-\left|\frac{f(c + h)}{h}\right| \leq \frac{f(c + h)}{h} \leq \left|\frac{f(c + h)}{h}\right|$$

we immediately conclude

$$\lim_{h\to 0} \frac{f(c+h)}{h} = 0.$$

That is, $f$ is differentiable at $x = c$ (and $f'(c) = 0$).