Exam 1 - Solutions

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Problem 1: Find $\int_{-2}^{3} 2x^2\,[\,|x|\,]\,dx$. (Here, as usual, $[x]$ denotes the largest integer $\leq x$.)

Solution Note that

$$2x^2\lfloor |x| \rfloor = \left\{ \begin{array}{c} 2x^2 \text{ for } -2 < x \leq -1 \\ 0 \text{ for } -1 < x < 1 \\ 2x^2 \text{ for } 1 \leq x < 2 \\ 4x^2 \text{ for } 2 \leq x < 3 \end{array} \right\}.$$

Hence,

$$\begin{aligned} \int_{-2}^{3} 2x^2\lfloor |x| \rfloor dx &= \int_{-2}^{-1} 2x^2 dx + \int_{1}^{2} 2x^2 dx + \int_{2}^{3} 4x^2 dx \\ &= \frac{2}{3}x^3\Big|_{-2}^{-1} + \frac{2}{3}x^3\Big|_{1}^{2} + \frac{4}{3}x^3\Big|_{2}^{3} \\ &= -\frac{2}{3} + \frac{16}{3} + \frac{16}{3} - \frac{2}{3} + \frac{108}{3} - \frac{32}{3} = \frac{104}{3}. \end{aligned}$$

Problem 2: Let f be an integrable function on $[a, b]$ and $a < d < b$. Further suppose that

$$\int_{a+d}^{b+d} f(x-d)dx = 4, \qquad \int_{-a}^{-d} f(-x)dx = 7.$$

Find

$$\int_{d}^{b} 2f(x)dx.$$

Solution Properties of the integral imply

$$\int_{a}^{b} f(x)dx = 4, \qquad \int_{a}^{d} f(x) = -7.$$

As $4 = \int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)dx = -7 + \int_d^b f(x)dx$, we see that $\int_d^b f(x)dx = 11$. Again, using properties of the integral, $\int_d^b 2f(x)dx = 2\int_d^b f(x)dx = 22$.

Problem 3: Suppose A, B are inductive sets. Prove $A \cap B$ is an inductive set. Give an example of inductive sets A, B such that $A - B$ is not an inductive set.

Solution If A and B are inductive sets, then $1 \in A, B$; thus, $1 \in A \cap B$. Moreover, suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$; hence, $x + 1 \in A$ and $x + 1 \in B$ because A and B are inductive sets. But, then $x + 1 \in A \cap B$. Therefore, $A \cap B$ is an inductive set.

Let $A = B = \mathbb{R}$. Then A and B are inductive sets because $1 \in \mathbb{R}$ and because $x \in \mathbb{R}$ implies $x + 1 \in \mathbb{R}$ by closure of addition for the real numbers. However, $A - B = \emptyset$ is not an inductive set since $1 \notin \emptyset$.

Problem 4: Let f be a bounded, integrable function on $[0, 1]$. Suppose there exists $C \in \mathbb{R}$ such that $f(x) \geq C > 0$ for all $x \in [0, 1]$. Prove that $g(x) = 1/f(x)$ is integrable on $[0, 1]$.

Solution Let $\epsilon > 0$ and observe that as f is integrable and $f \geq C$, there exist step functions $s(x), t(x)$ such that $C/2 \leq s(x) \leq f(x) \leq t(x)$ and $\int_0^1 (t(x) - s(x))dx < \epsilon \cdot C^2/4$. Let $s_1(x) = 1/t(x), t_1(x) = 1/s(x)$. Then, $0 < s_1(x) \leq g(x) \leq t_1(x)$ (we proved that in class on the first day). Moreover,

$$\int_0^1 (t_1(x) - s_1(x))dx = \int_0^1 \frac{1}{s(x)} - \frac{1}{t(x)} dx = \int_0^1 \frac{t(x) - s(x)}{s(x)t(x)} dx.$$

By choice, we have that $s(x), t(x) \geq C/2$. Thus, $s(x)t(x) \geq C^2/4$ and $1/(s(x)t(x)) \leq 4/C^2$. It follows that

$$\int_0^1 t_1(x) - s_1(x) dx \leq 4/C^2 \int_0^1 (t(x) - s(x))dx < 4/C^2 \epsilon \cdot C^2/4 = \epsilon.$$

Here the first inequality comes from the comparison principle for integrals of step functions and the second follows by hypothesis. Thus, by the Riemann condition, $g = 1/f$ is integrable.

Problem 5: Suppose f is defined for all $x \in (-1, 1)$ and that $\lim_{x \to 0} f(x) = A$. Show there exists a constant $c < 1$ such that $f(x)$ is bounded for all $x \in (-c, c)$.

Solution First, denote $f(0) = B$. Let $M = \max\{|B|, |A| + 1\}$. Since $\lim_{x\to 0} f(x) = A$, there exists $\delta > 0$ such that $|f(x) - A| < 1$ if $0 < |x| < \delta$. Thus, for all $0 < |x| < \delta$,

$$|f(x)| = |f(x) - A + A| \leq |f(x) - A| + |A| < 1 + |A|.$$

Now, set $c = \delta$. Then, for all $x \in (-c, c)$, $|f(x)| \leq M$.