Exam 2 Solutions

October 29, 2010

Total: 60 points

Problem 1: Find the derivative of each of the following functions

- $g(x) = \log(\cos(x^2))$
- $h(x) = e^{\sqrt{x}\sin x}$

Solution For the first problem, we invoke the chain rule twice. Thus,

$$\frac{d}{dx}(\log(\cos(x^2))) = \frac{1}{\cos(x^2)} \cdot (-\sin(x^2)) \cdot 2x = -2x\tan(x^2).$$

The second problem requires the chain rule and the product rule. We immediately get,

$$\frac{d}{dx}(e^{\sqrt{x}\sin x}) = e^{\sqrt{x}\sin x}\left(\frac{1}{2\sqrt{x}}\sin x + \sqrt{x}\cos x\right).$$

Problem 2: Consider the function

$$g(x) = \frac{\log x}{x^2}.$$

Determine the behavior of g in a neighborhood of $x = 1$. Specifically, is the function increasing or decreasing? Is it convex or concave? Justify your answers.

Solution We determined $\frac{d}{dx}\log x = \frac{1}{x}$ for $x > 0$ and $\frac{d}{dx}x^2 = 2x \neq 0$ for $x > 0$. Thus, we can use the quotient rule to determine $g'(x), g''(x)$ away from $x = 0$. A quick calculation yields

$$g'(x) = \frac{1 - 2\log x}{x^3}$$

and

$$g''(x) = \frac{-5 + 6\log x}{x^4}.$$

As $\log 1 = 0$, $g'(1) = 1 > 0, g''(1) = -5 < 0$. Further, observe that $g'(x), g''(x)$ are both continuous functions for all $x > 0$. Thus, by the sign preservation of continuous functions, in a neighborhood of $x = 1$ we know $g'(x) > 0, g''(x) < 0$. By the Mean Value Theorem, $g' > 0$ implies that g is increasing in a neighborhood of $x = 1$. By the second derivative test, $g'' < 0$ implies g is concave in a neighborhood of $x = 1$.

Problem 3: Consider the functions $f(x) = x \sin x$ and $g(x) = (x + 5)\cos x$. Prove there exists $c \in (0, \pi/2)$ such that $f(c) = g(c)$. (If you are using a theorem, make sure you explain why the function or functions you are considering satisfy the hypotheses of the theorem.)

Solution Set $h(x) = x \sin x - (x + 5)\cos x$. Notice that $x, x + 5, \sin x, \cos x$ are all continuous functions on $[0, \pi/2]$, so the products and sums are also continuous. That is, $h(x)$ is continuous on $[0, \pi/2]$. Further, observe $h(0) = -5$ and $h(\pi/2) = \pi/2$. As $-5 < 0 < \pi/2$, the intermediate value theorem guarantees the existence of $c \in (0, \pi/2)$ such that $h(c) = 0$. As $h(c) = f(c) - g(c)$, it follows that $f(c) = g(c)$.

Problem 4: Define $f(x)$ such that $f(x) = x$ for every rational value of x and $f(x) = -x$ for every irrational x.
(a) Prove $f(x)$ is continuous at $x = 0$.
(b) Set $a \neq 0$. Prove that $f(x)$ is not continuous at $x = a$.

Solution (a) First, observe that $f(0) = 0$ as 0 is rational. Now, let $\epsilon > 0$ and choose $\delta = \epsilon$. If $|x - 0| = |x| < \delta$ then $|f(x) - f(0)| = |f(x)| = |x| < \delta = \epsilon$. Therefore, $f(x)$ is continuous at $x = 0$.
Alternative solution: Observe that $-|x| \leq f(x) \leq |x|$. As $\lim_{x\to 0} \pm|x| = 0$, it follows that $\lim_{x\to 0} f(x) = 0 = f(0)$ by the squeeze theorem.
(b) Set $\epsilon = |a|$. We show that for any $\delta > 0$ there exists $x \in (a - \delta, a + \delta)$ such that $|f(x) - f(a)| > \epsilon$.

To that end, consider an arbitrary $\delta > 0$. Choose $\delta_1 < \min\{\delta, |a|/2\}$. Then any $x \in (a - \delta_1, a + \delta_1)$ has the same sign as a. Now, suppose $a \in \mathbb{Q}$. Then by the density of the irrationals, there exists $x \notin \mathbb{Q}$ such that $x \in (a - \delta_1, a + \delta_1) \subset (a - \delta, a + \delta)$. Further, $|f(a) - f(x)| = |a - (-x)| = |a + x| = |a| + |x| > \epsilon$. The last equality comes as a, x have the same sign and the inequality follows by our choice of ϵ. Thus, f is not continuous for all $a \neq 0$ such that a is rational.

An analogous proof works when $a \notin \mathbb{Q}$ by the density of the rationals. It follows that f is not continuous for all $x \neq 0$.

Problem 5: Let f be continuous. Prove that

$$\int_0^x f(t)(x-t)dt = \int_0^x \big(\int_0^t f(u)du \big) dt.$$

Solution Define the function $g(x) = \int_0^x f(t)(x-t)dt - \int_0^x \big(\int_0^t f(u)du \big) dt$. By the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\left(\int_0^x tf(t)dt \right) = xf(x), \quad \frac{d}{dx}\left(\int_0^x \int_0^t f(u)dudt \right) = \int_0^x f(u)du.$$

Notice here we needed both that f and the indefinite integral of f are continuous functions. Now,

$$g'(x) = \int_0^x f(t)dt + xf(x) - xf(x) - \int_0^x f(u)du = 0.$$

So by the Mean Value Theorem, g is a constant. Further, observe that

$$g(0) = \int_0^0 f(t)(x-t)dt - \int_0^0 \int_0^t f(u)dudt = 0 - 0 = 0.$$

Thus, $g(x) = 0$ for all x. It follows that $\int_0^x f(t)(x-t)dt = \int_0^x \big(\int_0^t f(u)du \big) dt$.