Exam 3 Solutions

Problem 1. Evaluate $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\log(x+1)}\right)$.

Solution We begin by writing the problem as a single fraction, $\lim_{x\to 0}\left(\frac{\log(x+1)-x}{x\log(x+1)}\right)$. Observe that both numerator and denominator have limit zero, and thus we can apply L'Hopital's rule to see

$$\lim_{x\to 0}\left(\frac{\log(x+1)-x}{x\log(x+1)}\right)=\lim_{x\to 0}\left(\frac{1/(x+1)-1}{\log(x+1)+x/(x+1)}\right)=\lim_{x\to 0}\left(\frac{1-(x+1)}{(x+1)\log(x+1)+x}\right).$$

Note that in the expression on the right, the limits of both numerator and denominator are again zero. Thus a second application of L'Hopital's rule gives

$$\lim_{x\to 0}\left(\frac{1-(x+1)}{(x+1)\log(x+1)+x}\right)=\lim x\to 0\left(\frac{-1}{\log(x+1)+(x+1)/(x+1)+1}\right)=-\frac{1}{2}.$$

Problem 2. Evaluate $\int \frac{3x-2}{x^2-6x+10}dx$.

Solution We start by observing that the denominator can be written as $(x-3)^2+1$. That makes part of the problem easy:

$$-2\int\frac{dx}{(x-3)^2+1}=-2\arctan(x-3).$$

For the other part of the problem, we make the substitution $x-3=u$. So $x=u+3$. And thus we integrate:

$$3\int\frac{u+3}{u^2+1}du=3\int\frac{u}{u^2+1}du+3\int\frac{3du}{u^2+1}=\frac{3}{2}\log(u^2+1)+9\arctan u.$$

Here the last equality comes from a simple substitution. After substituting $x-3=u$ and adding in our work above, we get

$$\frac{3}{2}\log((x-3)^2+1)+7\arctan(x-3)+C.$$

Problem 3: Let f be an infinitely differentiable function on $\mathbb{R}$. We say f is analytic on $(-1,1)$ if the sequence $\{T_nf(x)\}$ converges to $f(x)$ for all $x\in(-1,1)$, where $T_nf(x)$ is the nth Taylor polynomial of f centered at zero. Suppose there exists a constant $0<C\leq 1$ such that

$$\left|f^{(k)}(x)\right|\leq C^k k!$$

for every positive integer k and every real number $x \in (-1,1)$. Prove that f is analytic on $(-1,1)$.

Solution Recall $f(x) = T_n f(x) + E_n(x)$ where $|E_n(x)| \leq \frac{|x|^n f^{(n)}(c)}{n!}$ for some c between 0 and x. Applying our bound on $f^{(n)}(c)$, we have

$$|f(x) - T_n f(x)| = |E_n(x)| \leq |Cx|^n.$$

It follows that $\lim_{n\to\infty} |f(x) - T_n f(x)| \leq \lim_{n\to\infty} |Cx|^n$. Hence, as $|Cx| < 1$ for all $x \in (-1,1)$, $\lim_{n\to\infty} |Cx|^n = 0$. We conclude that $E_n f(x) \to 0$ and thus $T_n f(x) \to f(x)$; that is, f is analytic.

Problem 4: Let $f(x)$ be a function defined on $(0,\pi]$. Suppose $\lim_{n\to\infty} f(1/n) = 0$ and $\lim_{n\to\infty} f(\pi/n) = 1$. Prove that $\lim_{x\to 0^+} f(x)$ does not exist.

Solution Since $\lim_{n\to\infty} f(\frac{1}{n}) = 0$ and $\lim_{n\to\infty} f(\frac{\pi}{n}) = 1$, there exist positive integers $N_1 > 0$ and $N_2 > 0$ such that $n_1 > N_1$ implies $|f(\frac{1}{n})| < 1/4$ and $n_2 > N_2$ implies $|f(\frac{\pi}{n}) - 1| < 1/4$. Put $N = \max\{N_1, N_2\}$. If $n > N$, then $f(\frac{1}{n}) < \frac{1}{4}$, $f(\frac{\pi}{n}) > \frac{3}{4}$, and $f(\frac{\pi}{n}) - f(\frac{1}{n}) > \frac{1}{2}$.
Now suppose $\lim_{x\to 0^+} f(x)$ exists and is equal to the finite number L. Then there exists δ such that whenever $0 < x < \delta$, we have $|f(x) - L| < \frac{1}{4}$. Let $M > \max\{N, \frac{\pi}{\delta}\}$ be a positive integer. If $n > M$, then $0 < \frac{1}{n}, \frac{\pi}{n} < \delta$ and

$$\left| f\left(\frac{1}{n}\right) - f\left(\frac{\pi}{n}\right) \right| \leq \left| f\left(\frac{1}{n}\right) - L \right| + \left| f\left(\frac{\pi}{n}\right) - L \right| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

But, since $n > M > N$, this contradicts the conclusion of the first paragraph $f(\frac{\pi}{n}) - f(\frac{1}{n}) > \frac{1}{2}$. We conclude that $\lim_{x\to 0^+} f(x)$ does not exist.

Problem 5: A function f on $\mathbb{R}$ is compactly supported if there exists a constant $B > 0$ such that $f(x) = 0$ if $|x| \geq B$. If f and g are two differentiable, compactly supported functions on $\mathbb{R}$, then we define

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y) g(y) dy.$$

Note: We define $\int_{-\infty}^{\infty} f(x)dx = \lim_{t\to\infty} \int_{-t}^{t} f(x)dx$.

- Prove $(f * g)(x) = (g * f)(x)$.
- Prove $(f' * g)(x) = (g' * f)(x)$.

Solution To be done on Pset 11!!

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