PRACTICE PROBLEMS FOR THE FINAL EXAM

(1) Determine each limit, if it exists:
(a) $\lim_{x\to\infty} \frac{x\sin(1/x)}{\cos(\pi/2+1/x)}$
(b) $\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin(3x)}$
(c) $\lim_{x\to 0} \frac{x\sin(x^2)}{2\log(x^3+1)}$. (Hint: Use Taylor approximations rather than L'Hopital here. You'll save 20 minutes.)

(2) Evaluate each integral or find an antiderivative:
(a) $\int x\sin x\cos x dx$
(b) $\int \frac{x+1}{(x^2+2x+2)^3} dx$
(c) $\int_{-1}^{1} x^{-1/5} dx$
(d) $\int \sin^3 x dx$
(e) $\int_0^\infty \frac{dx}{\sqrt{x}}$

(3) Determine whether the following series converge absolutely, converge conditionally, or diverge:
(a) $\sum \frac{1}{(\log n)^5}$
(b) $\sum \frac{(n!)^2}{2^{n^2}}$
(c) $\sum \frac{3^n}{n^n}$
(d) $\sum \frac{(-1)^n}{\log n}$
(e) $\sum \frac{(-1)^n 5n^2}{n^3+10}$

(4) Determine the radius of convergence for each of the following series:
(a) $\sum \frac{x^n n^n}{n!}$
(b) $\sum \frac{2^n x^n}{2n}$
(c) $\sum \frac{(n!)^2}{(2n)!} x^n$

(5) Using power series already familiar to you from class, determine the power series each of the following functions. Also determine the radius of convergence.
(a) $f(x) = \frac{x}{(1+x)^2}$
(b) $g(x) = \cosh x = \frac{e^x + e^{-x}}{2}$

(6) Compute the derivative of $\sqrt{x}$ directly from the definition of the derivative.

(7) Prove the following statement by induction:
$(1 + 2 + \cdots + n)^2 = 1^3 + \cdots n^3$.

(8) Which of the following functions is integrable on the interval $[-1, 1]$? Justify why or why not.
(a) $f(x) = x^2$.
(b) Let $g(x) = 1$ if the decimal expansion of x contains a zero, and let $g(x) = 0$ if the decimal expansion of x does not contain a zero.
(c) Let $g(x) = x\sin(1/x)$ if $x \neq 0$, and let $g(x) = 0$ if $x = 0$.

(9) The statement below is an incorrect statement of the Riemann condition:
- A function f defined on $[a,b]$ is integrable on $[a,b]$
  if and only if
- there exists $\epsilon \in \mathbb{R}^+$ such that for all step functions s, t on $[a,b]$ we have $\int_a^b (t-s) < \epsilon$.

Prove that the first statement does not imply the second. Then give the correct statement of the Riemann condition.

(10) Let $f(x)$ be a continuous function with continuous first derivative such that $f(0) = 0$ and $0 \leq f(x) \leq e^{\alpha x}$ where $0 < \alpha < 1$. Prove that

$$\int_0^\infty f'(x)e^{-x}dx = \int_0^\infty f(x)e^{-x}dx.$$

(11) Let $f(x) = \int_0^x \frac{t}{1+t}e^{-t}dt$. Prove that $\lim_{x\to\infty} f(x)$ exists and is bounded above by 1.

(12) Let $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$, where f is an integrable function defined on $[0,1]$. Prove $\int_0^1 |f(x)|dx \leq ||f||_\infty$.

(13) Prove $|\int f| \leq \int |f|$ for f integrable.

(14) A set $A \subset \mathbb{R}$ is called *open* if for each $x \in A$ there exists some $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset A$. Let f be a continuous function on $\mathbb{R}$. Prove $S = \{x | f(x) > 0\}$ is open.

(15) Suppose f is a differentiable function on $(0,1)$ and f' is bounded on $(0,1)$. Prove f is bounded on $(0,1)$.

(16) We know $\lim_{n\to\infty} x^n = f(x)$ on $[0,1]$ where $f(x) = 0$ for $x \in [0,1)$ and $f(1) = 1$. Prove the convergence is NOT uniform. (Do not use the fact that the limit is discontinuous.)

(17) Given a sequence $\{a_n\}$ consider a sequence of positive integers $\{n_k\}$ such that $n_1 < n_2 < \cdots$. We call $\{a_{n_k}\}$ a *subsequence* of $\{a_n\}$. Suppose $\{a_{n_k}\}$ and $\{a_{n_l}\}$ are two different subsequences of $\{a_n\}$ such that $\lim_{k\to\infty} a_{n_k} \neq \lim_{l\to\infty} a_{n_l}$. Prove $\lim_{n\to\infty} a_n$ diverges.

(18) Let $f_n(x) = \sin\left(\frac{x}{n}\right)$. For any fixed $R \in \mathbb{R}^+$, prove $f_n(x)$ converges uniformly to $f(x) = 0$ on $[-R, R]$.

(19) Suppose the series $\sum a_n x^n$ converges absolutely for $x = -4$. What can you say about the radius of convergence? Does the series $\sum n|a_n|2^n$ converge?

(20) Suppose the series $\sum_{n=1}^\infty a_n$ converges absolutely. Prove $\sum_{n=1}^\infty (e^{a_n} - 1)$ converges absolutely.

(21) Assume $\sum a_n$ converges and $\{b_n\}$ is a bounded sequence. Prove $\sum a_n b_n$ converges.