Practice Exam 1 - Solutions

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Problem 1: Compute $\int_{99}^{103}(2x-198)^2 \left[\sqrt{x-99}\,\right] dx$ where here $[x]$ is defined to be the largest integer $\leq x$.

Solution By the properties of the integral, we know that the above is equal to

$$4\int_0^4 (x)^2 \left[\sqrt{x}\,\right] dx.$$

Now, $[\sqrt{x}\,]$ takes the value 0 on $[0,1)$ and the value 1 on $[1,4)$ and the value 2 at $x=4$. Thus, we can rewrite the integral

$$4\int_1^4 x^2 dx + 4\int_4^4 x^2 dx = 4\frac{4^3}{3} - 4\frac{1^3}{3} = 256/3 - 4/3 = 252/3 = 84.$$

Problem 2: Let S be a square pyramid with base area r^2 and height h. Using Cavalieri's Theorem, determine the volume of the pyramid.

Solution Orient the square pyramid so that the base sits on the $x-y$ plane and the top vertex sits on the z axis. Let $a_S(h_0)$ denote the cross-sectional area of $S \cap \{z = h_0\}$, and note that this is a square that will be a function of r, h. To find the length of a side of the square at height h_0, we consider the line which contains the two points $(r, 0)$ and $(0, h)$. One form of the equation for this line is $y = \frac{-h}{r}x + h$. Notice that here x is the side length of the square at height y. Thus, for $y = h_0$ we get $x = \frac{r}{h}(h - h_0)$. That is, $a_S(h_0) = \frac{r^2}{h^2}(h-h_0)^2$ for $h_0 \in [0, h]$ and $a_S(h_0) = 0$ otherwise. So

$$v(S) = \int_0^h \frac{r^2}{h^2}(h-h_0)^2\, dh_0 = \frac{r^2}{h^2}\int_0^h h^2 - 2hh_0 + h_0^2\, dh_0$$

$$= \frac{r^2}{h^2}(h^2(h-0) - 2h(h^2/2) + h^3/3) = r^2h/3.$$

Problem 3: Let f be an integrable function on $[0,1]$. Prove that $|f|$ is integrable on $[0,1]$.

Solution Let $\epsilon > 0$. The Riemann condition implies there exist step functions $s(x), t(x)$ on $[0,1]$ such that $s(x) \leq f(x) \leq t(x)$ and $\int_0^1 (t(x)-s(x))dx <$

ϵ. Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[0,1]$ such that $s(x), t(x)$ are both constant (s_k, t_k on the k-th) open subintervals of P. Denote $A \subset P$, such that $A = \{x \in [0,1] | s_k(x) \geq 0 \text{ or } t_k(x) \leq 0\}$.

Choose $s_1(x)$ such that $s_1(x) = \min\{|s(x)|, |t(x)|\}$ for $x \in A$ and $s_1(x) = 0$ for $x \in P - A$. Choose $t_1(x) = \max\{|s(x)|, |t(x)|\}$. Denote by $(s_1)_k$ the constant value taken by $s_1(x)$ on the k-th subinterval. Define $(t_1)_k$ similarly. It is straightforward to check, using properties of the absolute value, that $s_1(x) \leq |f(x)| \leq t_1(x)$ on $[0,1]$.
Now consider any open subinterval (x_{k-1}, x_k). If $s_k \geq 0$ then $(t_1)_k - (s_1)_k = t_k - s_k$ and if $t_k \leq 0$ then $(t_1)_k - (s_1)_k = -s_k - (-t_k) = t_k - s_k$. Finally, observe that if $s_k < 0, t_k > 0$ then $(t_1)_k - (s_1)_k = \max\{|t_k|, |s_k|\} < |t_k| + |s_k| = t_k - s_k$.

Thus,

$$\int_0^1 t_1(x) - s_1(x) dx = \sum_{k=1}^{n} ((t_1)_k - (s_1)_k)(x_k - x_{k-1})$$

$$\leq \sum_{k=1}^{n} (t_k - s_k)(x_k - x_{k-1}) = \int_0^1 t(x) - s(x) dx < \epsilon.$$

It follows that $|f|$ is integrable.

Problem 4: The well ordering principle states that every non-empty subset of the natural numbers has a least element. Prove the well ordering principle implies the principle of mathematical induction. (Hint: Let $S \subset \mathbb{P}$ be a set such that $1 \in S$ and if $k \in S$ then $k+1 \in S$. Consider $T = \mathbb{P} - S$. Show that $T = \emptyset$.)

Solution Let $S \subset \mathbb{P}$ such that $1 \in S$ and if $k \in S$ then $k+1 \in S$. We wish to show $S = \mathbb{P}$. Suppose not. Then there exists a set $T = \mathbb{P} - S$, and by hypothesis T is non-empty. Moreover, as $T \subset \mathbb{P}$, the well ordering principle implies that T has a least element. Let m denote this least element. By Theorem 1 in Course Notes A, $m \geq 1$. Notice $1 \notin T$ as $1 \in S, 1 \in \mathbb{P}$, and thus $m > 1$. As m is the least element of T, $m - 1 \notin T$.

Claim: $m \in \mathbb{P}$ with $m > 1$ implies $m - 1 \in \mathbb{P}$.
Proof of claim: Suppose not. Then there exists an inductive set A such that $m - 1 \notin A$. Consider the set $B = A - \{m\}$. We wish to prove B is inductive. Since $m \neq 1$, $1 \in B$. Now suppose $k \in B \subset A$. As A is inductive, $k + 1 \in A$

and thus $k+1 \in B$ or $k+1 = m$. In the first case, B is inductive. The second case implies $m-1 = k \in B \subset A$. This contradicts the construction of A. Thus, if $m-1 \notin \mathbb{P}$, then there exists an inductive set B such that $m \notin B$. This contradicts the assumption that $m \in \mathbb{P}$. It follows that $m-1 \in \mathbb{P}$.

Now, back to the main proof. As $m-1 \notin T$ and $m-1 \in \mathbb{P}$ it follows that $m-1 \in S$. But by the properties of S, $m-1+1 = m \in S$. This contradicts the fact that $m \in T$. It follows that T cannot have a least element. The well ordering principle then implies that $T = \emptyset$. That is $S = \mathbb{P}$.

Problem 5: Suppose $\lim_{x\to p^+} f(x) = \lim_{x\to p^-} f(x) = A$. Prove $\lim_{x\to p} f(x) = A$

Solution Let $\epsilon > 0$. By hypothesis, there exist δ_1, δ_2 such that $p < x < p+\delta_1$ implies $|f(x) - A| < \epsilon$ and $p - \delta_2 < x < p$ implies $|f(x) - A| < \epsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - p| < \delta$ implies $|f(x) - A| < \epsilon$.