Practice Exam 2 Solutions

Problem 1. Find

$$\lim_{h\to 0} \frac{\int_0^{1+h} e^{t^2}\,dt - \int_0^1 e^{t^2}\,dt}{h(3+h^2)}.$$

Solution First, using that the limit of a product is the product of limits, we get

$$\lim_{h\to 0} \frac{\int_0^{1+h} e^{t^2}\,dt - \int_0^1 e^{t^2}\,dt}{h(3+h^2)} = \lim_{h\to 0} \frac{\int_0^{1+h} e^{t^2}\,dt - \int_0^1 e^{t^2}\,dt}{h} \lim_{h\to 0} \frac{1}{3+h^2}.$$

Because $\frac{1}{3+h^2}$ is a continuous function at $h=0$, the second limit is $\frac{1}{3}$. Define

$$g(x) = \int_0^x e^{t^2}\,dt.$$

Then the first limit is

$$g'(1) = \lim_{h\to 0} \frac{g(1+h)-g(1)}{h}.$$

By the fundamental theorem of calculus, $g'(1) = e^{1^2} = e$. Mutiplying the two limits together, our final answer is $\frac{e}{3}$.

Problem 2. Find $(f^{-1})'(0)$ where $f(x) = \int_0^x \cos(\sin(t))dt$ is defined on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Solution First, we check that f is strictly increasing and continuous on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. To show f is stricly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, it is enough to show $f'(x) > 0$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. By the fundamental theorem of calculus, $f'(x) = \cos(\sin(x))$. For $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have $\sin(x) \in (-1,1)$, and for $y \in (-1,1)$, we have $\cos y > 0$. Thus, f is strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Moreover, f is continuous on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ by theorem 3.4 and differentiable on $(-\frac{\pi}{2}, \frac{\pi}{2})$ by the fundamental theorem of calculus (theorem 5.1). Then by theorem 6.7,

$$(f^{-1})'(f(0)) = \frac{1}{f'(0)}.$$

Since $f(0) = 0$, and $f'(0) = \cos(\sin(0)) = 1$ by the fundamental theorem of calculus (theorem 5.1), we deduce $(f^{-1})'(0) = 1$.

Problem 3: In each case below, assume f is continuous for all x. Find $f(2)$.
(a)

$$\int_0^x f(t)dt = x^2(1+x).$$

(b)

$$\int_0^{f(x)} t^2 dt = x^2(1+x).$$

Solution (a) Differentiating both sides of the equality yields

$$f(x) = 2x(1+x) + x^2 = 3x^2 + 2x.$$

To differentiate the left hand side, we used the fundamental theorem of calculus. To differentiate the right hand side, we used the product rule. Plugging in $x = 2$ yields

$$f(2) = 16.$$

(b) For this part, we integrate the left hand side to get

$$\frac{f(x)^3}{3} = x^2(1+x).$$

Plugging in $x = 2$ and solving for $f(2)$, we get $f(2) = (36)^{\frac{1}{3}}$.

Problem 4. Give an example of a function $f(x)$ defined on $[-1,1]$ such that

- f is continuous and differentiable on $[-1,1]$.
- f' is not continuous for at least one value $x \in [-1,1]$.

Solution Let

$$f(x) = \left\{ \begin{array}{ll} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array} \right\}.$$

For $x \neq 0$, $f(x)$ is a product of compositions of differentiable functions. Thus, f is differentiable for $x \in [-1,1]$, $x \neq 0$. Note

$$0 \leq \left| \frac{h^2 \sin(\frac{1}{h})}{h} \right| \leq \left| \frac{h^2}{h} \right| = |h|.$$

Thus, using the squeezing principle (theorem 3.3), we deduce

$$\lim_{h\to 0}\left|\frac{f(h)-f(0)}{h}\right| = 0.$$

We conclude that $f'(0)$ exists and equals zero. By a theorem from class, f is continuous on $[-1,1]$ because f is differentiable on $[-1,1]$. Next, we need to show that f' is discontinuous at $x=0$. By the product rule and the chain rule, we have

$$f'(x) = 2x\sin(1/x) - \cos(1/x) \text{ if } x \neq 0,$$

and we know $f'(0)=0$. Assume f' is continuous at $x=0$. Then there must exist $\delta > 0$ such that $|x| < \delta$ implies that $|f'(x)| < \frac{1}{2}$. Choose $x_0 = \frac{1}{2\pi n} < \delta$ with n a positive integer. Then

$$f'(x_0) = 2\frac{1}{2\pi n}\sin(2\pi n) - \cos(2\pi n) = 0 - 1 = -1.$$

But, by assumption $|f'(x_0)| < \frac{1}{2}$. This is a contradiction. Thus, f' is not continuous at $x=0$.

Problem 5. Let $f(x)$ be continuous on $[0,1]$, and assume $f(0)=f(1)$. Show that for any $n \in \mathbb{Z}^+$, there exists at least one $x \in [0,1]$ such that $f(x) = f(x+\frac{1}{n})$.

Solution Consider the continuous function $g_n(x) = f(x) - f(x+\frac{1}{n})$ on the interval $[0, \frac{n-1}{n}]$. Consider the set $g_n(0), g_n(\frac{1}{n}), \ldots, g_n(\frac{n-1}{n})$. If $g_n(\frac{k}{n}) = 0$ for some k then $f(x+\frac{k}{n}) = f(x+\frac{k}{n}+\frac{1}{n})$ and we are done. Hence, we may assume that $g_n(\frac{k}{n}) \neq 0$ for $k = 0, 1, \ldots n-1$. Note

$$\sum_{k=0}^{n-1} g_n\left(\frac{k}{n}\right) = f(0) - f(1) = 0.$$

If $g_n(\frac{k}{n}) > 0$ for $k = 0, \ldots, n-1$, then the sum is positive, and if $g_n(\frac{k}{n}) < 0$ for $k = 0, \ldots, n-1$, then the sum is negative. Since the sum is neither positive nor negative, there must be k_1 and k_2 such that $g_n(\frac{k_1}{n}) > 0$ and $g_n(\frac{k_2}{n}) < 0$. Putting $y_1 = \min\{\frac{k_1}{n}, \frac{k_2}{n}\}$ and $y_2 = \max\{\frac{k_1}{n}, \frac{k_2}{n}\}$, we note that $g_n(y_1)$ and $g_n(y_2)$ have opposite signs. Therefore, by the intermediate value theorem, there must be $y \in (y_1, y_2)$ such that $g_n(y) = 0$. In particular,

$$f(y) = f\left(y + \frac{1}{n}\right)$$

as desired.