

Separation of Variables

We'll now look at the technique we used to solve the previous problem and discuss what other sorts of problems that method is useful for.

In general, the method of separation of variables applies to differential equations that can be written as:

$$\frac{dy}{dx} = f(x)g(y).$$

In our previous example, $f(x) = -x$ and $g(y) = y$.

The key step in the method is the separation of variables. It is possible because Leibnitz designed his notation to make this work nicely, allowing us to treat differential calculations as if they were ordinary arithmetic.

$$\begin{aligned}\frac{dy}{dx} &= f(x)g(y) \\ \frac{dy}{g(y)} &= f(x) dx \quad \text{which we can write as} \\ h(y) dy &= f(x) dx \quad \text{where } h(y) = \frac{1}{g(y)}.\end{aligned}$$

Next we antidifferentiate both sides of the equation:

$$H(y) = \int h(y) dy; \quad F(x) = \int f(x) dx$$

In our example, $H(y) = \ln |y|$ and $F(x) = \frac{-x^2}{2}$.

These antiderivatives are equal, so we get:

$$H(y) = F(x) + c \quad (\text{Again, we only need one constant } c.)$$

This is what we call an *implicit equation*; in our example we had $\ln y = -x^2/2 + c$. It doesn't tell us exactly what y is but it does describe y implicitly. In order to solve for y explicitly we need the inverse function H^{-1} ; in our example this inverse was the exponential function and the explicit equation was $y = Ae^{-x^2/2}$.

In practice, it's often easy to find the implicit equation and quite messy to perform the inverse operation. In that case we might leave the solution in implicit form.

Remark 1: In the example, we could have written $\ln |y| = -x^2/2 + c$ for $y \neq 0$. Then we would have gotten $|y| = Ae^{-x^2/2}$ or $y = \pm Ae^{-x^2/2}$, which is almost exactly the answer we did get, if $a = \pm A$ and $A > 0$.

Professor Jerison didn't bother with this because it makes the calculation more complicated. Once he had solved the problem for $y > 0$, he knew from previous experience what the answer would be and he could skip directly to that and check his work. (The exponential function comes up all the time, so you too will want to be completely comfortable dealing with it.)

This still leaves out the case $y = 0$. This is an extremely boring solution, but it is still a solution to this problem. You can verify that $y = 0$ (i.e. $a = 0$) is a solution to this problem by plugging 0 in for y in the equation $(\frac{d}{dx} + x)y = 0$ or $\frac{dy}{dx} = -xy$.

It's not so surprising that we missed that solution, because in the process of separating variables we divided by y . If you divide by something you may have problems when that thing equals zero, or miss that solution. To avoid these problems, take note of when you divide by something that might be zero and double check that case after you've finished your calculations.

Remark 2: We had:

$$\int h(y) dy = \int f(x) dx$$

which evaluated to:

$$H(y) = F(x) + c.$$

We could have written $H(y) + c_1 = F(x) + c_2$, but this is equivalent to $H(y) = F(x) + c_2 - c_1 = F(x) + c$. To save time and writing, we write down only one arbitrary constant when integrating both sides of an equation.

Remark 3: In our example, the additive constant c turned into a multiplicative constant A when we calculated $e^{-x^2/2+c}$. In general there will always be a free constant in the solution to a differential equation, but that constant will not always be additive.

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