Proof of the First Fundamental Theorem of Calculus

The first fundamental theorem says that the integral of the derivative is the function; or, more precisely, that it's the difference between two outputs of that function.

Theorem: (First Fundamental Theorem of Calculus) If f is continuous and $F' = f$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Proof: By using Riemann sums, we will define an antiderivative G of f and then use $G(x)$ to calculate $F(b) - F(a)$.

We start with the fact that $F' = f$ and f is continuous. (It's not strictly necessary for f to be continuous, but without this assumption we can't use the second fundamental theorem in our proof.)

Next, we define $G(x) = \int_a^x f(t)\,dt$. (We know that this function exists because we can define it using Riemann sums.)

The second fundamental theorem of calculus tells us that:

$$G'(x) = f(x)$$

So $F'(x) = G'(x)$. Therefore,

$$(F - G)' = F' - G' = f - f = 0$$

Earlier, we used the mean value theorem to show that if two functions have the same derivative then they differ only by a constant, so $F - G =$ constant or

$$F(x) = G(x) + c.$$

This is an essential step in an essential proof; all of calculus is founded on the fact that if two functions have the same derivative, they differ by a constant.

Now we compute $F(b) - F(a)$ to see that it is equal to the definite integral:

$$\begin{aligned}
F(b) - F(a) &= (G(b) + c) - (G(a) + c) \\
&= G(b) - G(a) \\
&= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\
&= \int_a^b f(t)\,dt - 0 \\
F(b) - F(a) &= \int_a^b f(x)\,dx
\end{aligned}$$