Logs and Exponents

a) Prove that for $x > 1$:

$$a \int_{1/x}^{1} \frac{1}{t}\,dt = \int_{(1/x)^a}^{1} \frac{1}{t}\,dt.$$

b) Assume $x > 1$. What is the geometric interpretation of the result of part a?

c) What does this tell you about the area between the x-axis and the graph of $\frac{1}{x}$ over the interval from 0 to 1?

Solution

a) Prove that for $x > 1$:

$$a \int_{1/x}^{1} \frac{1}{t}\,dt = \int_{(1/x)^a}^{1} \frac{1}{t}\,dt.$$

Don't be intimidated by the word "prove" — to solve this problem we need only evaluate two definite integrals and remark that they're equal!

$$\begin{aligned} a \int_{1/x}^{1} \frac{1}{t}\,dt &= a \ln(t)\big|_{1/x}^{1} \\ &= a(\ln(1) - \ln(1/x)) \\ &= -a\ln(1/x) = a\ln(x) \end{aligned}$$

$$\begin{aligned} \int_{(1/x)^a}^{1} \frac{1}{t}\,dt &= \ln(t)\big|_{(1/x)^a}^{1} \\ &= \ln(1) - \ln((1/x)^a) \\ &= -a\ln(1/x) = a\ln(x) \end{aligned}$$

By evaluating the definite integrals we see that their values are equal.

b) Assume $x > 1$. What is the geometric interpretation of the result of part a?

Because $\int_c^1 \frac{1}{t}\,dt$ equals the area between the graph of $y = \frac{1}{t}$ and the t-axis over the interval from c to 1, our answer to (a) tells us:

> The area under the graph of $\frac{1}{t}$ between $(1/x)^a$ and 1 is a times as large as the area under the graph of $\frac{1}{t}$ between $1/x$ and 1.

c) What does this tell you about the area between the x-axis and the graph of $\frac{1}{x}$ over the interval from 0 to 1?

As a grows large, $(1/x)^a$ approaches 0. Even if the area under the graph of $\frac{1}{t}$ between $1/x$ and 1 is very small, the value of a times that area will go to infinity as a does. In other words, the area between the graph of $\frac{1}{x}$ and the x-axis over the interval between 0 and 1 is infinite.

We will explore this further in our discussion of indefinite integrals, near the end of the course.