Volume of a Spheroid

The solid of revolution generated by rotating (either half of) the region bounded by the curves $x^2+4y^2 = 4$ and $x = 0$ about the y-axis is an example of an oblate spheroid. Compute its volume.

Solution

We could calculate the volume using shells or disks. The equation describing x as a function of y is slightly simpler than that describing y as a function of x, so we'll integrate with respect to y and use disks.

First, we solve for x:

$$\begin{aligned} x^2 + 4y^2 &= 4 \\ x^2 &= 4 - 4y^2 \\ x &= \pm\sqrt{4 - 4y^2} \\ x &= \pm 2\sqrt{1 - y^2}. \end{aligned}$$

We're told we can use either half of the region, so we'll choose $x = 2\sqrt{1 - y^2}$.

Next we determine the limits of integration. If we're familiar with ellipses, we know that $(0, 1)$ and $(0, -1)$ are the highest and lowest points on the ellipse. If not, we can at least observe that the expression describing x is undefined when $|y| > 1$. Hence our limits of integration are $y = -1$ and $y = 1$.

Our integral sums the volumes of disks with radius $2\sqrt{1 - y^2}$ and height dy:

$$\begin{aligned} \int_{-1}^{1} \pi(2\sqrt{1 - y^2})^2\, dy &= 4\pi \int_{-1}^{1} 1 - y^2\, dy \\ &= 4\pi \left[y - \frac{y^3}{3} \right]_{-1}^{1} \\ &= 4\pi \left[\left(1 - \frac{1}{3}\right) - \left(-1 - \left(-\frac{1}{3}\right)\right) \right] \\ &= \frac{16\pi}{3} \end{aligned}$$

This is two thirds of the volume of a cylinder containing the spheroid, so is probably correct.