## Volume of Revolution Via Washers

**Problem:** By integrating with respect to the variable $y$, find the volume of the solid of revolution formed by rotating the region bounded by $y = 0$, $x = 4$ and $y = \sqrt{x}$ about the line $x = 6$.

**Solution:** This problem was solved in recitation using the shell method. Here we use the washer method.

First we sketch the region; see Figure 1.

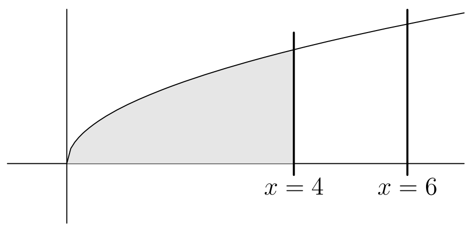


Figure 1: Region to be revolved.

This region is to be rotated about the line $x = 6$. The result will be a donut shaped volume; since the region we're rotating doesn't extend all the way to the line $x = 6$ the volume of revolution will have a hole in the middle.

The simplest way to compute the volume of this solid is by using shells and integrating with respect to $x$; this was described in the recitation video. It's also possible to compute the volume by slicing the solid horizontally into washer shaped pieces and summing the volumes of the washers.

Each washer will have a height $dy$, an inner radius $r$, and an outer radius $R \geq r$. In this example the inner radius is determined by the distance between the line $x = 4$ and the line $x = 6$, so $r = 2$. The outer radius $R$ is the horizontal distance between the graph of $y = \sqrt{x}$ and the line $x = 6$. This distance is a function of $y$; it's large (6) at the base of the donut shape and small (2) at the top.

Since we'll be integrating with respect to the variable $y$ we'll want to solve $y = \sqrt{x}$ for $x$ in terms of $y$.

$$x = y^2$$

The outer radius is the (positive) distance between this $x$ position and $x = 6$, so $R = 6 - y^2$.

The volume of each washer is the volume of a disk of radius $R$ that's had a disk of radius $r$ removed from its center.

$$\begin{aligned}
dV &= (\text{volume of disk of radius } R) - (\text{volume of disk of radius } r) \\
&= \pi R^2 dy - \pi r^2 dy \\
&= \pi(R^2 - r^2) dy \\
&= \pi((6 - y^2)^2 - (2)^2) dy \\
&= \pi(36 - 12y^2 + y^4 - 4) dy
\end{aligned}$$

$$dV = \pi(32 - 12y^2 + y^4)dy$$

The volume of the solid is given by a Riemann sum of the volumes of the washers; i.e. by an integral. The base of the volume is at $y = 0$ and its highest point is where the curve $y = \sqrt{x}$ meets the line $x = 4$, so it has height $2 = \sqrt{4}$.

$$\begin{aligned}
\text{Volume} &= \int_{y=0}^{y=2} dV \\
&= \int_0^2 \pi(32 - 12y^2 + y^4 - 4)dy \\
&= \pi \left(32y - 12\frac{y^3}{3} + \frac{y^5}{5})\right|_0^2 \\
&= \pi \left(32 \cdot 2 - 4 \cdot 2^3 + \frac{2^5}{5}\right) - 0 \\
&= \frac{192}{5}\pi
\end{aligned}$$

To check our work we note that this is close to $36\pi$, which is very roughly the volume of rotation of a washer with cross sectional area 9 and radius 2.