Average Value

You already know how to take the average of a finite set of numbers:

$$\frac{a_1 + a_2}{2} \text{ or } \frac{a_1 + a_2 + a_3}{3}$$

If we want to find the average value of a function $y = f(x)$ on an interval, we can average several values of that function:

$$\text{Average} \approx \frac{y_1 + y_2 + ... + y_n}{n}.$$

As was mentioned previously, if we let the number of values n approach infinity we get:

$$\text{Continuous Average} = \frac{1}{b-a}\int_a^b f(x)\,dx = \text{Ave}(f).$$

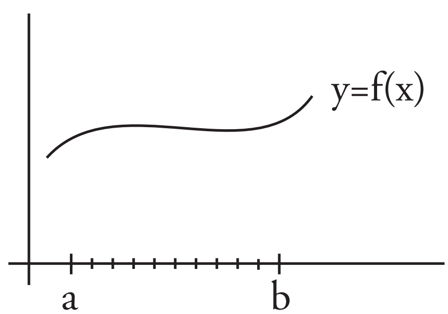

Figure 1: $a \leq x \leq b$.

Why does this describe the average value of $f(x)$? Imagine that you have $n+1$ equally spaced points $a = x_0 < x_1 < x_2 < ... < x_n = b$. The distance between each pair of points is $\Delta x = \frac{b-a}{n}$. Let $y_0 = f(x_0)$, $y_1 = f(x_1), ...y_n = f(x_n)$.

Then the Riemann sum approximating the area under the curve is:

$$(y_1 + y_2 + \cdots + y_n)\Delta x.$$

As n approaches infinity this approaches the area under the curve, which is:

$$\int_a^b f(x)\,dx.$$

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &\approx \frac{1}{b-a}(y_1+y_2+\cdots+y_n)\Delta x \\ &= \frac{1}{b-a}(y_1+y_2+\cdots+y_n)\frac{b-a}{n} \\ &= \frac{y_1+y_2+\cdots+y_n}{n},\end{aligned}$$

so:

$$\frac{1}{b-a}\int_a^b f(x)\,dx \approx \frac{y_1+y_2+\cdots+y_n}{n}.$$

The only difference between the average value and the integral (area under the curve) is that we're dividing by the length of the interval.

Example: Find the average value of $f(x) = c$ on the interval $[a, b]$, where a, b and c are arbitrary constants.

$$\begin{aligned}\frac{1}{b-a}\int_a^b c\,dx &= \frac{1}{b-a}\cdot(\text{Area of a } (b-a) \text{ by } c \text{ rectangle}) \\ &= \frac{1}{b-a}\cdot(b-a)\cdot c \\ &= c\end{aligned}$$

If the value of $f(x)$ is always c, then the average value of $f(x)$ had better be c. This confirms that our formula for the average value of a function works, and in particular it confirms that $\frac{1}{b-a}$ is the correct normalizing factor. In this case our Riemann sum becomes:

$$\begin{aligned}\frac{y_1+y_2+\cdots+y_n}{n} &= \frac{\overbrace{c+c+\cdots+c}^{n\text{times}}}{n} \\ &= \frac{nc}{n} \\ &= c\end{aligned}$$

and we see why we needed the n in the denominator.