Average Bank Balance

An amount of money A_0 compounded continuously at interest rate r increases according to the law:

$$A(t) = A_0 e^{rt} \qquad (t\text{=time in years.})$$

a) What is the average amount of money in the bank over the course of T years?

b) Check your work by plugging in $A_0 = \$100$, $r = .05$ and $T = 1$; does the result seem plausible?

Solution

a) What is the average amount of money in the bank over the course of T years?

The average value of a function $f(x)$ over the interval $[a, b]$ is:

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

In our example, the function is $A(t) = A_0 e^{rt}$ and the interval is $[0, T]$. Noting that the antiderivative of e^{rt} is $\frac{1}{r}e^{rt}$, we find:

$$\begin{aligned}
\text{Avg}(A) &= \frac{1}{T-0}\int_0^T A_0 e^{rt}\,dt \\
&= A_0 \frac{1}{rT}\, e^{rt}\Big|_0^T \\
&= \frac{A_0}{rT}\left(e^{rT} - e^0\right) \\
\text{Avg}(A) &= \frac{A_0}{rT}\,(e^{rT} - 1).
\end{aligned}$$

This is the difference between the final and initial balance, divided by rate times time!

b) Check your work by plugging in $A_0 = \$100$, $r = .05$ and $T = 1$; does the result seem plausible?

If $r = .05$ and $T = 1$ then:

$$\begin{aligned}
\text{Avg}(A) &= \frac{\$100}{.05}(e^{.05} - 1) \\
&= \$102.5
\end{aligned}$$

This seems plausible because if we were dealing with simple interest (rather than continuously compounded interest), our starting balance would be \$100 and our final balance would be \$105.