

## Boiling Cauldron, Continued

Last class we asked how much energy it would take to boil water in a cauldron whose shape is found by rotating a parabola with height 1 meter and width 2 meters about the  $y$ -axis. The approximately 1600 liters of water would start at a uniform  $T = 0$  degrees Celsius. We assumed that the final temperature would be given by the formula

$$T = 100 - 30y \text{ degrees Celsius.}$$

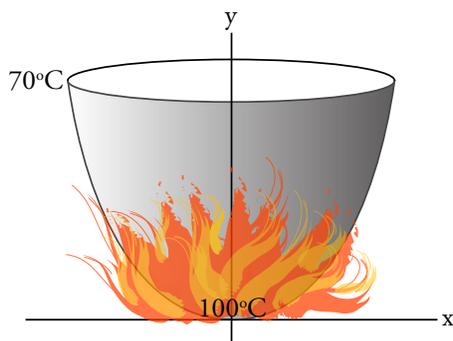


Figure 1: The boiling cauldron.

It's important to realize  $T$  is constant on each horizontal level and so the method of disks is the best way to add up the energy needed. If we set up the integral with shells,  $T$  would vary from top to bottom of the shell and we'd have to solve a second calculus problem to figure out how much heat was needed for that shell.

The equation of the parabola is  $y = x^2$ , so each disk will have radius  $x = \sqrt{y}$  and height  $dy$ . To calculate the amount of energy needed we'll have to add up

$$\text{energy} = \text{degrees} \cdot \text{volume.}$$

So our answer will be:

$$\int_0^1 \underbrace{(100 - 30y)}_{\text{temperature}} \underbrace{\pi x^2 dy}_{\text{disk volume}}$$

$$\begin{aligned} \int_0^1 (100 - 30y)\pi y dy &= \int_0^1 (100y - 30y^2)\pi dy \\ &= [50\pi y^2 - 10\pi y^3]_0^1 \\ &= 50\pi - 10\pi \\ &= 40\pi \text{ deg} \cdot \text{m}^3 \end{aligned}$$

Our answer has units of degrees Celsius times cubic meters. Let's translate that into something more familiar.

$$\begin{aligned} 40\pi \text{ deg} \cdot \text{m}^3 \left( \frac{1 \text{ cal}}{\text{deg} \cdot \text{cm}^3} \right) \left( \frac{100 \text{ cm}}{\text{m}} \right)^3 &= 40\pi \cdot 10^6 \text{ calories} \\ &= 40\pi \cdot 1000 \text{ Kcal} \\ &\approx 125,000\pi \text{ Kcal} \end{aligned}$$

One candy bar has about 250 Kcal, so it takes the energy of about 500 candy bars to heat the cauldron.

**Question:** What does this integral give us?

**Answer:** We must heat each milliliter of water in the cauldron up to some temperature between 70 and 100 degrees Celsius. For each milliliter there's some amount of energy needed to do that — water lower down in the pot needs to be heated to a higher temperature, and so will require more energy. A calorie is the amount of energy needed to raise one cubic centimeter of water by one degree Celsius. This integral adds up the energy needed to heat every single drop of water in the cauldron to exactly the right temperature.

Before we go on, let's compute the average final temperature. (The average initial temperature is 0 because the temperature is initially constant.) The formula for a weighted average is

$$\frac{\int_a^b f(x)w(x) dx}{\int_a^b w(x) dx}.$$

In this example, our function  $f(x)$  is the temperature  $T$  and our weight  $w(y) = \pi y$  corresponds to the volume of water in a horizontal disk; the denominator is the total volume of water in the cauldron. The limits of integration are still 0 and 1.

$$\begin{aligned} \frac{\int_0^1 T\pi y dy}{\int_0^1 \pi y dy} &= \frac{40\pi}{\pi/2} \\ &= 80 \text{ degrees} \end{aligned}$$

The value of the weight function  $w(y)$  is different at different heights. This makes sense; there's more water at the top of the cauldron than at the bottom. If you tried to take the average the ordinary way you would get:

$$\frac{T_{\max} + T_{\min}}{2} = \frac{100 + 70}{2} = 85 \text{ degrees.}$$

This estimate is higher than the weighted average because it doesn't take into account that there is more water at the top of the cauldron (70 degrees) than at the bottom (100 degrees).

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