Weighted Average

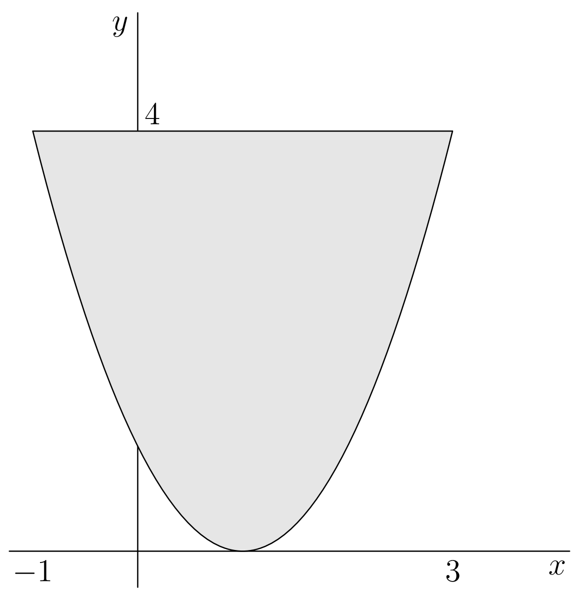
The centroid or center of mass of a planar region is the point at which that region balances perfectly, like a plate on the end of a stick. The coordinates of the centroid are given by weighted averages.

The x coordinate of the centroid is $\bar{x} = \frac{\int x\, dA}{\int dA}$, where dA is an infintessimal portion of area; the weighting function in this average is just x.

Similarly, the y coordinate of the centroid is $\bar{y} = \frac{\int y\, dA}{\int dA}$.

Find the centroid $(\bar{x}, \bar{y})$ of the parabolic region bounded by $x = -1$, $x = 3$, $y = (x-1)^2$ and $y = 4$.

Solution



We might guess from the symmetry of the region that $\bar{x} = 1$. We'll do the computation for practice. Since we're integrating with respect to x, our area dA will be a vertical rectangle with base at $y = (x-1)^2$, top at $y = 4$ and width dx.

$$\bar{x} = \frac{\int_{-1}^{3} x(4-(x-1)^2)\, dx}{\int_{-1}^{3} (4-(x-1)^2)\, dx}$$

$$\begin{aligned}
\int_{-1}^{3} x(4-(x-1)^2)\, dx &= \int_{-1}^{3} 3x - x^3 + 2x^2\, dx \\
&= \left[\frac{3}{2}x^2 - \frac{1}{4}x^4 + \frac{2}{3}x^3\right]_{-1}^{3}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{3}{2}\cdot 9 - \frac{1}{4}\cdot 81 + \frac{2}{3}\cdot 27\right) - \left(\frac{3}{2} - \frac{1}{4} - \frac{2}{3}\right) \\
&= 10\frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
\int_{-1}^{3} (4-(x-1)^2)\,dx &= \int_{-1}^{3} 3 - x^2 + 2x\,dx \\
&= \left[3x - \frac{1}{3}x^3 + x^2\right]_{-1}^{3} \\
&= \left(3\cdot 3 - \frac{1}{3}\cdot 27 + 9\right) - \left(-3 + \frac{1}{3} + 1\right) \\
&= 10\frac{2}{3}
\end{aligned}
$$

We conclude that $\bar{x} = \frac{10\frac{2}{3}}{10\frac{2}{3}} = 1$.

To find $\bar{y}$ we will be integrating with respect to y. Our area dA will have width equal to the width of the region and height dy. The left side of the region has the equation $x = 1 - \sqrt{y}$ and the right side has equation $x = 1 + \sqrt{y}$, so the width is $(1+\sqrt{y}) - (1-\sqrt{y}) = 2\sqrt{y}$.

$$
\bar{y} = \frac{\int_0^4 y(2\sqrt{y})\,dy}{\int_0^4 2\sqrt{y}\,dy}
$$

$$
\begin{aligned}
\int_0^4 y(2\sqrt{y})\,dy &= 2\int_0^4 y^{3/2}\,dy \\
&= 2\cdot \frac{2}{5}y^{5/2}\Big|_0^4 \\
&= \frac{4}{5}(2^5 - 0) \\
&= \frac{2^7}{5}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^4 2\sqrt{y}\,dy &= 2\int_0^4 y^{1/2}\,dy \\
&= 2\cdot \frac{2}{3}y^{3/2}\Big|_0^4 \\
&= \frac{4}{3}\cdot(2^3 - 0) \\
&= \frac{2^5}{3}
\end{aligned}
$$

Hence, $\bar{y} = \frac{\frac{2^7}{5}}{\frac{2^5}{3}} = \frac{12}{5} = 2.4.$

The center of mass of the parabolic region is at $(1, 2.4)$; this seems plausible given that the top of the parabola ($y = 4$) will weigh more than the bottom ($y = 0$).