

## Area Under the Bell Curve

Today, we'll complete the calculation first mentioned during the discussion of the fundamental theorem of calculus. We've said that:

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Now we'll prove it. The proof relies on a very clever trick which we would be unlikely to come up with ourselves. We study the proof because the result is very important and because it's related to adding up slices, as we've been doing in this unit.

In our example with the little brother and the dart board, we found the volume of revolution created by rotating the curve  $e^{-r^2}$  around the vertical axis.

The calculation went by shells:

$$V = \int_0^{\infty} 2\pi r e^{-r^2} dr$$

where  $2\pi r$  was the circumference of the shell,  $e^{-r^2}$  was the height, and  $dr$  was the thickness.

$$\begin{aligned} V &= \int_0^{\infty} 2\pi r e^{-r^2} dr \\ &= -\pi e^{-r^2} \Big|_0^{\infty} \\ V &= \pi \end{aligned}$$

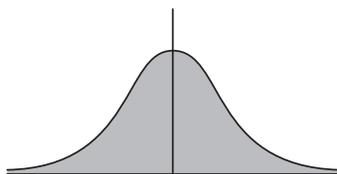


Figure 1:  $Q = \text{area under } e^{-t^2}$ .

What we need to find now is not a volume but an area:

$$Q = \int_{-\infty}^{\infty} e^{-t^2} dt$$

This is the area under the bell curve shown in Figure 1.

The trick is to compute  $V$  in a different way — by slices. Amazingly, we'll discover that  $V = Q^2$ , which will tell us the value of  $Q$ :

$$\begin{aligned} Q^2 &= V \\ Q^2 &= \pi \\ Q &= \sqrt{\pi}. \end{aligned}$$

So now we need to check that  $V = Q^2$ .

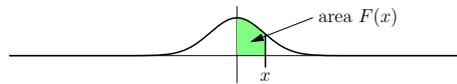


Figure 2:  $F(x) = \int_0^x e^{-t^2} dt$ .

Remember that we have already discussed:

$$F(x) = \int_0^x e^{-t^2} dt.$$

We were interested in the limit of  $F(x)$  as  $x$  approached infinity:

$$F(\infty) = \int_0^\infty e^{-t^2} dt.$$

That's the area under half of the bell curve, so:

$$\begin{aligned} Q &= 2F(\infty) \\ F(\infty) &= \frac{\sqrt{\pi}}{2} \end{aligned}$$

By calculating the value of  $Q$  we're reassuring ourselves that this is true.

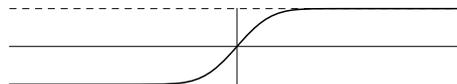


Figure 3:  $\lim_{x \rightarrow \infty} F(x) = \frac{\sqrt{\pi}}{2}$

We need to prove  $V = Q^2$  by using slices; two slices of the surface  $e^{-r^2}$  are shown in Figure 4. This is not particularly easy to visualize. It may help to imagine slicing an anthill, a pile of gravel, or some other bump that has circular symmetry.

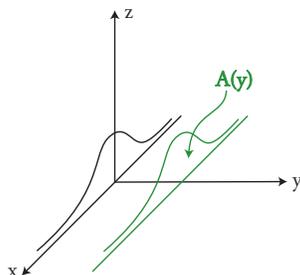


Figure 4: Three dimensional slices of the volume of rotation of  $e^{-r^2}$ .

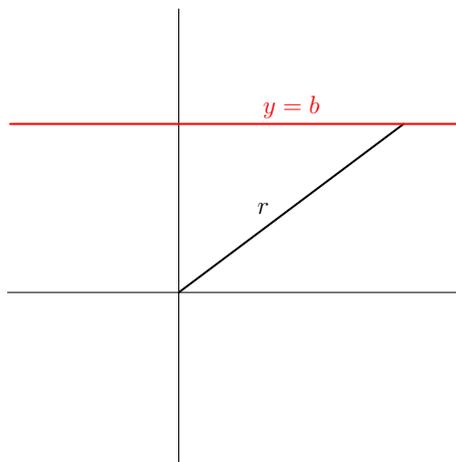


Figure 5: Top view of a slice of the surface of revolution of  $e^{-r^2}$ .

The formula for volume by slices is:

$$V = \int_{-\infty}^{\infty} A(y) dy.$$

We're going to fix  $y = b$  and calculate  $A(b)$ .

Figure 5 shows a top view of the slice whose area we're calculating. The height of the surface at a point  $r$  units away from  $(0, 0)$  is given by:

$$\text{height} = e^{-r^2}.$$

In terms of  $b$  and  $x$ ,  $r^2 = b^2 + x^2$ , so

$$\begin{aligned} \text{height} &= e^{-(b^2+x^2)} \\ &= e^{-b^2} e^{-x^2} \\ &= ce^{-x^2} \end{aligned}$$

where  $c$  is a constant equal to  $e^{-b^2}$ .

The area under the curve is:

$$\begin{aligned} A(b) &= \int_{-\infty}^{\infty} e^{-b^2} e^{-x^2} dx \\ &= e^{-b^2} \int_{-\infty}^{\infty} e^{-x^2} dx \\ A(b) &= e^{-b^2} Q \end{aligned}$$

We've calculated the area of a slice of our "bump" in terms of the value  $Q$  that we're looking for.

Remember that the volume of that bump is given by:

$$\begin{aligned} V &= \int_{-\infty}^{\infty} A(y) dy \\ &= \int_{-\infty}^{\infty} e^{-y^2} Q dy \\ &= Q \int_{-\infty}^{\infty} e^{-y^2} dy \quad (Q \text{ is a constant}) \\ &= Q^2 \quad (\text{by definition, } Q = \int_{-\infty}^{\infty} e^{-t^2} dt). \end{aligned}$$

As desired, we've proven that  $V = Q^2$  and so we can conclude that  $Q = \sqrt{\pi}$ .

**Question:** Doesn't  $x$  change as  $y$  changes?

**Answer:** Good question. That's the way  $x$  and  $y$  have been used in this whole course. But  $x$  and  $y$  can be different variables; they won't always depend on each other.

In this example, we're fixing  $y = b$ . The value of  $y$  doesn't change. The value of  $x$  does change;  $x$  varies from negative infinity to positive infinity. In this example  $y$  is not a function of  $x$ , and that's ok but it's not what we're used to.

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