Integrate by Partial Fractions

Use the method of partial fractions to compute the integral:

$$\int \frac{x^2 + 2x + 3}{(x+1)(x+2)(x+3)}\, dx.$$

Solution

We first check to see if we can factor the numerator to cancel any terms in the denominator; we can't. Since all the terms in the denominator are linear, we need not try to factor them. The numerator is a second degree polynomial and the denominator is third degree, so we do not need to perform any long division of polynomials.

We set up the partial fractions decomposition as follows:

$$\frac{x^2 + 2x + 3}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}.$$

Now we apply the cover-up method, starting by solving for A:

$$\frac{x^2 + 2x + 3}{\cancel{(x+1)}(x+2)(x+3)} = \frac{A}{\cancel{x+1}} \cancel{+ \frac{B}{x+2} + \frac{C}{x+3}}.$$

We cover up $x+1$ and any expression that does not contain $x+1$, then plug in $x = -1$; the value that makes $x+1$ equal 0. We get:

$$\frac{(-1)^2 + 2 \cdot (-1) + 3}{(-1+2)(-1+3)} = 1 = A.$$

We repeat this process for B and C:

$$\begin{aligned}
\frac{x^2 + 2x + 3}{(x+1)\cancel{(x+2)}(x+3)} &= \cancel{\frac{A}{x+1}} + \frac{B}{\cancel{x+2}} + \cancel{\frac{C}{x+3}} \\
\frac{(-2)^2 + 2(-2) + 3}{(-2+1)(-2+3)} &= B \\
B &= -3
\end{aligned}$$

$$\begin{aligned}
\frac{x^2 + 2x + 3}{(x+1)(x+2)\cancel{(x+3)}} &= \cancel{\frac{A}{x+1} + \frac{B}{x+2}} + \frac{C}{\cancel{x+3}} \\
\frac{(-3)^2 + 2(-3) + 3}{(-3+1} -3+2 &= C \\
C &= 3
\end{aligned}$$

We conclude that:

$$\frac{x^2 + 2x + 3}{(x+1)(x+2)(x+3)} = \frac{1}{x+1} - \frac{3}{x+2} + \frac{3}{x+3}.$$

If we plug in $x = 0$ we get $\frac{1}{2} = 1 - \frac{3}{2} + 1$, so this is probably a correct decomposition.

We can now break this down into three relatively simple integrals. One integration is presented in detail below, using the substitution $u = x + 3$; the other two are similar:

$$\begin{aligned} \int \frac{3}{x+3}\, dx &= 3 \int \frac{1}{u}\, du \\ &= 3 \ln |u| + c \\ &= 3 \ln |x+3| + c \end{aligned}$$

In conclusion,

$$\begin{aligned} \int \frac{x^2 + 2x + 3}{(x+1)(x+2)(x+3)}\, dx &= \int \frac{1}{x+1} - \frac{3}{x+2} + \frac{3}{x+3}\, dx \\ &= \ln |x+1| - 3 \ln |x+2| + 3 \ln |x+3| + c \end{aligned}$$

We could try to check this by differentiation, but that leads directly to verifying that our partial fractions decomposition is correct — a time consuming task.

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