Graph of $r = 2a\cos\theta$

Let's get some more practice in graphing and polar coordinates. We just found the area enclosed by the curve $r = 2a\cos\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. What happens when θ doesn't lie in this range?

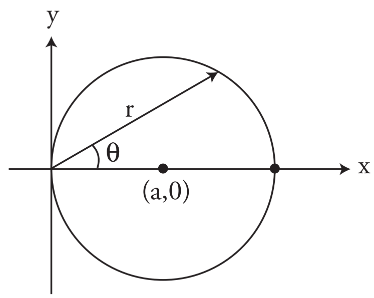

Figure 1: Off center circle $r = 2a\cos\theta$.

When $\frac{\pi}{2} < \theta < \pi$, r is negative. For example, when $\theta = \frac{3\pi}{4}$, $\cos\theta = -\frac{\sqrt{2}}{2}$ and $r = -a\sqrt{2}$. If we move a distance of *negative* $a\sqrt{2}$ in the direction of angle $\frac{3\pi}{4}$ we arrive at the point $(-a\sqrt{2}, \frac{3\pi}{4})$, which is $(a, -a)$ in rectangular coordinates.

In fact, because we know that the points on the curve must have the property:

$$(x - a)^2 + y^2 = a^2$$

in rectangular coordinates, we know that as θ increases, the point $(2a\cos\theta, \theta)$ must remain on that same curve. As θ ranges from 0 to 2π (or from $-\frac{\pi}{2}$ to $\frac{3\pi}{2}$), the point $(2a\cos\theta, \theta)$ travels around the circle twice.

A common mistake is to choose the wrong limits of integration and count the same area twice, or cancel a positive area with an overlapping negative one.

Question: Can you find the area using the limits of integration 0 and π?

Answer: Yes. The integral $\displaystyle\int_0^\pi \frac{1}{2}(2a\cos\theta)^2\,d\theta$ gives a correct answer.

However, $r = 2a\cos\theta$, $0 \le \theta \le \pi$ is an awkward way to describe a circle. As θ ranges from 0 to $\frac{\pi}{2}$, r is positive and (r, θ) moves along the top half of the circle. As θ sweeps through the second quadrant ($\frac{\pi}{2} < \theta < \pi$), r is negative and so the curve appears in the fourth quadrant.

When we work with negative values of r it's easy to get confused, so when possible it's a good idea to choose our limits of integration so that r is positive.