Polar Coordinates and Conic Sections

Suppose we want to graph the curve described by:

$$r = \frac{1}{1 + 2\cos\theta}.$$

Again we start by plotting some points on this curve:

θ	r
0	$\frac{1}{3}$
$\frac{\pi}{2}$	-1
$-\frac{\pi}{2}$	1

By using the equations:

$$x = r\cos\theta, \qquad y = r\sin\theta$$

we can convert these polar coordinates to rectangular coordinates, show in Figure 1. For example, when $\theta = \frac{\pi}{2}$ we know that $r = 1$ and so:

$$\begin{aligned} x = r\cos\theta &= 0 \\ y = r\sin\theta &= -1 \end{aligned}$$

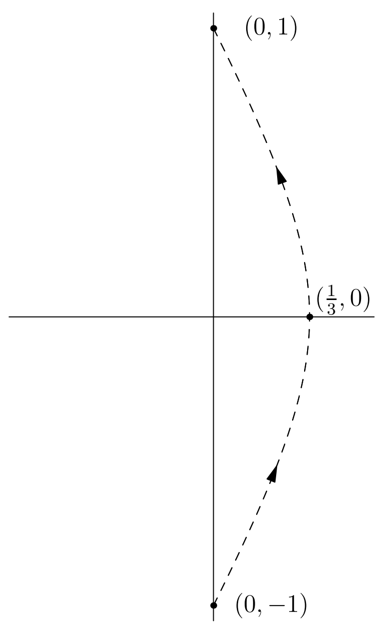

Figure 1: Rectangular coordinates of points on the curve $r = \dfrac{1}{1 + 2\cos\theta}$.

When $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, the denominator $1 + 2\cos\theta$ is positive and so r is positive; the curve traced over this interval must look something like the dotted line in Figure 1.

It's possible for the denominator to be 0:

$$\begin{aligned}1 + 2\cos\theta &= 0\\ 2\cos\theta &= -1\\ \cos\theta &= -\frac{1}{2}\\ \theta &= \arccos\left(-\frac{1}{2}\right)\\ \theta &= \pm\frac{2\pi}{3}\end{aligned}$$

The radius r goes to infinity as θ approaches $\frac{2\pi}{3}$ and $-\frac{2\pi}{3}$, so the curve will extend infinitely far in those directions.

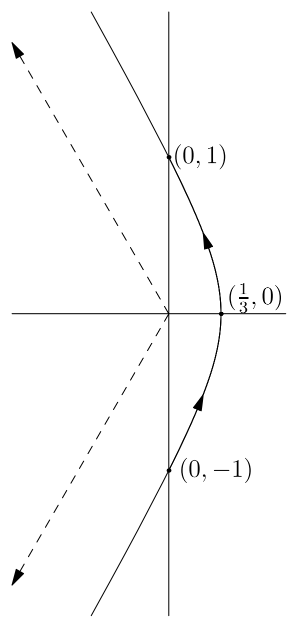

Figure 2: Graph of the curve $r = \dfrac{1}{1 + 2\cos\theta}$.

This is as much as we'll be able to figure out about the graph without converting its equation from polar to rectangular coordinates. Let's do that.

Rectangular Equation

What is the rectangular (x, y) equation for $r = \dfrac{1}{1 + 2\cos\theta}$?

To answer this question we could use our formula $x = r\cos\theta$, $y = r\sin\theta$ and then try to simplify, but if we're clever we can manipulate our original formula until it appears in terms of x and y.

$$\begin{aligned} r &= \frac{1}{1+2\cos\theta} \\ r + 2r\cos\theta &= 1 \\ r &= 1 - 2r\cos\theta \\ r &= 1 - 2x \end{aligned}$$

Multiplying both sides by the denominator simplified the expression and allowed us to make the substitution $x = r\cos\theta$. The variable θ no longer appears.

If we square both sides of this new equation we can get rid of the variable r as well:

$$\begin{aligned} r &= 1 - 2x \\ r^2 &= (1-2x)^2 \\ x^2 + y^2 &= 1 - 4x + 4x^2 \\ -3x^2 + y^2 + 4x - 1 &= 0 \end{aligned}$$

This is a standard calculation with a standard result; whenever we start with have $\dfrac{1}{a + b\cos\theta}$ or $\dfrac{1}{a + b\sin\theta}$ we'll end up with an equation like this.

Because the signs of the coefficients of x^2 and y^2 are different, this is the equation of a hyperbola. (If the signs match, the equation describes an ellipse; if one of these coefficients is 0 the graph is a parabola.) We can now conclude that the dotted lines in Figure 1 are asymptotes of the graph.

To complete our understanding of the curve $r = \frac{1}{1+2\cos\theta}$ we ask what happens when the denominator $1 + 2\cos\theta$ is negative?

Since we know that the equation for the curve in rectangular coordinates is $-3x^2 + y^2 + 4x - 1 = 0$, we can guess that for $\frac{3\pi}{2} < \theta < \frac{5\pi}{2}$ the curve must trace out the other branch of the hyperbola.

Connection to Kepler's Second Law

There is a beautiful connection between the basic formula for area and these types of curve.

As you may know, the trajectories of comets are hyperbolas. Ellipses are the trajectories of planets or asteroids. When you represent hyperbolas and ellipses in polar coordinates like this, it turns out that:

$$r = 0 \qquad \text{is the focus of the hyperbola.}$$

Polar coordinates are the natural way to express the trajectory of a planet or comet if you want the center of gravity (the sun) to be at the origin.

The formula

$$dA = \frac{1}{2} r^2 \, d\theta$$

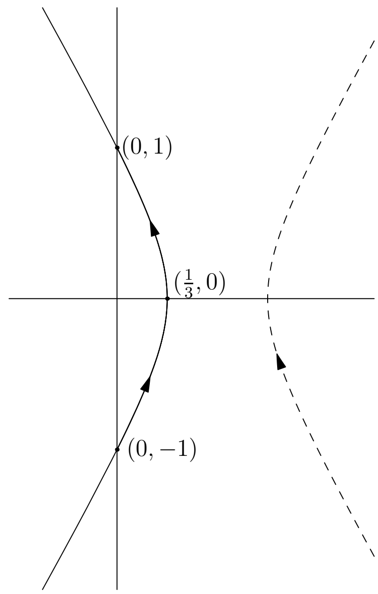

Figure 3: Both branches of the curve $r = \dfrac{1}{1 + 2\cos\theta}$.

is a central formula in astronomy. Kepler's second law says that a line joining a comet or planet to the sun sweeps out equal areas in equal time periods. In other words, the rate of change of area swept out is constant:

$$\frac{dA}{dt} = \text{constant}.$$

This tells us that as a comet travels around the sun, its speed varies, and varies predictably. Since we know $dA = \frac{1}{2}r^2\,d\theta$, we can conclude that:

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}.$$

Combining this with Kepler's second law we get:

$$\frac{1}{2}r^2\frac{d\theta}{dt} = \text{constant}.$$

In modern-day terms, what this formula says is that angular momentum is conserved. The objects Kepler was observing weren't subject to friction or air resistance, but this equation is the same one used to describe why a top keeps spinning after you let it go, or why an ice skater spins faster when she pulls her arms and legs in.