

Repeating L'Hôpital's Rule

This example illustrates the superiority of Version 1 of l'Hôpital's rule; it works even if $g'(a) = 0$.

In this case, $f(x) = \cos x - 1$, $g(x) = x^2$, and $a = 0$. We're trying to find:

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2}.$$

We can easily verify that $f(a) = g(a) = 0$.

We apply l'Hôpital's rule:

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2} = \lim_{x \rightarrow 0} \frac{-\sin x}{2x}.$$

Notice that $\frac{-\sin x}{2x}$ is undefined when $x = 0$; it's of the type $\frac{0}{0}$. That's OK; this version of l'Hôpital's rule still works when $g'(x) = 0$; it works as long as $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ is defined.

We need to find $\lim_{x \rightarrow 0} \frac{-\sin x}{2x}$. We can do that by applying l'Hôpital's rule!

$$\lim_{x \rightarrow 0} \frac{-\sin x}{2x} = \lim_{x \rightarrow 0} \frac{-\cos x}{2}.$$

All together, the calculation looks like:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2} &= \lim_{x \rightarrow 0} \frac{-\sin x}{2x} && \text{(l'Hop)} \\ &= \lim_{x \rightarrow 0} \frac{-\cos x}{2} && \text{(l'Hop)} \\ &= \frac{-\cos 0}{2} \\ &= -\frac{1}{2}. \end{aligned}$$

Notice that we only know that the hypotheses of l'Hôpital's rule hold for our problem when we reach the end and get the limit. The theorem says that l'Hôpital's rule only works if the limit exists, and we find out that the limit exists by using l'Hôpital's rule to calculate it. This is logically somewhat subtle, but in practice the theorem works very well.

Question: Why does the limit have to exist? Isn't it just the derivative that has to exist?

Answer: No, we need the derivative of the numerator, the derivative of the denominator *and* the limit to exist.

Surprisingly enough, we don't need $f'(a)$ to exist; we're working with limits as x approaches a , so what happens when $x = a$ doesn't necessarily matter to

us. Once again we're using limits to get close to something that's not defined at the exact value $x = a$.

If any one of the three limits $\lim_{x \rightarrow a} f'(x)$, $\lim_{x \rightarrow a} g'(x)$, and $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ does not exist then we can't apply l'Hôpital's rule. This is because the theorem we used to prove the rule works might not be true if these limits are undefined.

To get a little ahead of ourselves, notice that in:

$$\lim_{x \rightarrow \infty} x e^{-x}$$

the limit $\lim_{x \rightarrow \infty} x$ is undefined. Nevertheless, l'Hôpital's rule will apply here.

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