

**PROFESSOR:** Hi. Welcome back to recitation. In lecture, you learned about some new tools for computing certain limits. In particular, you learned about l'Hopital's rule. So now you have the ability to compute easily some limits that before might have been difficult to do. So I have here four examples of limits, some of which-- or maybe all of which, or maybe none of which or, well definitely some of which-- will be, you could use l'Hopital's rule on. So here they are. Why do you pause the video and take a few minutes to work them all out, come back, and we can work them out together.

Welcome back. So we have these four limits here. And let's start solving them, shall we? So I'll just go from the first one, and then I'll, you know, go through them in order. So limit (a) is the limit as  $x$  goes to 1 of  $x$  to the  $a$  minus 1 over  $x$  to the  $b$  minus 1. If we want to use l'Hopital's rule, we have to first check that this is a limit of an appropriate kind of expression. So that's, in particular, a limit that is a 0 over 0 or infinity over infinity form, so an indeterminate quotient.

And as  $x$  goes to 1, we see that  $x$  to the  $a$  goes to 1. So  $x$  to the  $a$  minus 1 goes to 0. And similarly  $x$  to the  $b$  goes to 1, so  $x$  to the  $b$  minus 1 goes to 0. So indeed, this is a 0 over 0 indeterminate form, so we can apply l'Hopital's rule. So we apply l'Hopital's rule. And it says that the limit as  $x$  goes to 1 of  $x$  to the  $a$  minus 1 over  $x$  to the  $b$  minus 1. OK, so what do we do? What l'Hopital's rule says is we can take the derivative of the top and the derivative of the bottom, and then the limit of the left side will be equal to the limit of the result provided the result actually has an existing limit, either a real number or infinity or minus infinity.

So in this case, this says that this expression is equal to the limit-- so it's the same limit-- as  $x$  goes to 1 of-- OK, so the derivative of the top is  $a x$  to the  $a$  minus 1, and the derivative of the bottom is  $b x$  to the  $b$  minus 1. So l'Hopital's rule says that these two limits are equal provided that the second limit exists.

So what is the second limit? Well, we-- OK, we can do some simplifications here. Here there wasn't, you know, a lot of obvious simplifying to be done. But here there's an obvious simplification step. So this is equal to the limit as  $x$  goes to 1-- so  $a$  over  $b$  is just a constant, and now the powers of  $x$ , this is  $x$  to the  $a$  minus  $b$ . But as  $x$  goes to 1,  $x$  to the  $a$  minus  $b$  goes to 1 no matter what  $a$  and  $b$  are. So at this point, this isn't an indeterminate form anymore. This is nice simple limit we can just plug in. So this is just equal to  $a$  over  $b$ . OK so that's part a, straightforward application of l'Hopital's rule.

Let's go look at part b. So part b asks us for the limit as  $x$  goes to 0 of  $\sin 5x$  over  $x$ . So I'll put that right here, maybe I'll draw a little border. So the limit as  $x$  goes to 0 of  $\sin 5x$  divided by  $x$ . So again there's a question of, you know, is this limit, do we know what it is already? And the answer is, maybe we do and maybe we don't, but suppose we don't. So in that case, can we apply l'Hopital's rule? Well we check, and as  $x$  goes to 0,  $\sin 5x$  goes to 0, and  $x$  goes to 0. So this is another 0 over 0 form. So we can apply l'Hopital's rule.

So l'Hopital's rule says that this is equal to the limit as  $x$  goes to 0 of-- we take the derivative of the top and divide by the derivative of the bottom. So the derivative of the top is just  $5 \cos 5x$ . And the derivative of the bottom is 1. So l'Hopital's rule says the limit here is equal to the limit here provided this limit exists. And now as  $x$  goes to 0, well, OK,  $5 \cos 0$  is constant. So we're just looking at the limit of  $5 \cos 0$  here. As  $x$  goes to 0, that just approaches 5 times  $\cos 0$ , which is 1. So this is equal to-- sorry, the  $\cos 0$  part is 1. So it's 5 times 1, which is 5.

Now I should note that we didn't actually really need l'Hopital's rule for this one. One thing you can remember is that this limit is actually the definition of the derivative of  $\sin 5x$  at  $x = 0$ . So what we just did with l'Hopital's rule was use the derivative to compute the derivative, if you like. So we used the fact that we already knew the derivative in order to compute it. We could also just say by definition, this is the derivative of  $\sin 5x$  at 0, which is  $5 \cos 0$ , which is 1, which is 5, rather. OK so that's part (b).

Now part (c) asks for the limit as  $x$  goes to 0 of  $x^2 - 6x + 2$ -- did I do that right, yes-- over  $x + 1$ . So this limit, you can't apply l'Hopital's rule? Why can't you apply l'Hopital's rule? Well, because it's not an indeterminate form. This limit is really easy to compute. You can just plug in  $x = 0$  here. The top is going to 2, and the bottom is going to 1. So the whole limit is going to 2 divided by 1, which is 2. So there's no-- not only, you don't need l'Hopital's rule, and also you can't apply l'Hopital's rule. This is not a form in which l'Hopital's rule applies. It's not an indeterminate form, OK, so, but OK, but the result is very easy to compute. So that's 2.

And we've got our last one now which is (d), the limit as  $x$  goes to infinity of  $\ln(1 + e^{3x})$  divided by  $2x + 5$ . OK, well is it obvious what this limit is? Not to me. So OK, so then we can check. It's a quotient. So can we apply l'Hopital's rule? Well we can apply l'Hopital's rule if it's an indeterminate quotient. So what's happening is  $x$  goes to infinity?

Well, let's do the bottom first, because that's easy. As  $x$  goes to infinity,  $2x$  plus  $5$  goes to infinity. And, OK, as  $x$  goes to infinity,  $e$  to the  $3x$  goes to infinity. So  $1$  plus  $e$  to the  $3x$  goes to infinity. So  $\log$  of  $1$  plus  $3$  to the  $3x$  goes to infinity. So this is an infinity over infinity form. That's an indeterminate form, an indeterminate quotient. So we can apply l'Hopital's rule. So l'Hopital's rule says that this is equal to the limit of the quotient of the derivatives, provided that that limit exists. So this is equal to the limit as  $x$  goes to infinity of--

OK, so we need to look at the ratio of the derivatives. So the bottom one is easy. That's  $2$ . For the top one, to compute this derivative, we need to use the chain rule here. So you take the derivative of this, and you get, well, a  $\log$ . So on the bottom we get  $1$  plus  $e$  to the  $3x$ . And then by the chain rule, up top you get  $3e$  to the  $3x$ . Or, just rewriting this a little bit in a nicer form, this is the limit as  $x$  goes to infinity of  $3$  over  $2$  times  $e$  to the  $3x$  over  $1$  plus  $e$  to the  $3x$ .

All right, so now the question is what do we do from here? Well is this limit obvious? Well, OK, so there are two situations here. One is, you might look at this and you might already know what this limit is. The reason you might know that, is that  $x$  goes to infinity,  $e$  to the  $3x$  is going to infinity, and  $1$  plus  $e$  to the  $3x$  is going to infinity. So this is an infinity over infinity indeterminate form. And so you could apply l'Hopital's rule to it again. OK? That's one thing you could do.

You might also look at and say well, these two things are going to infinity at exactly the same rate. Right? This plus  $1$  is almost totally irrelevant. Actually for purposes of estimating the magnitude of this fraction, it is totally irrelevant. When you take-- you know, when  $3x$  is big, then  $e$  to the  $3x$  is like a billion or something. So  $e$  to  $3x$  is a billion. So you have a billion over a billion and  $1$ . So it's very, very close to  $1$ . It's going to get closer and closer to  $1$ .

So, OK, so if you didn't trust that analysis based on magnitudes, I mean the idea here is just that the most significant term in the top is the  $e$  to the  $3x$ . The most significant term on the bottom is  $e$  to the  $3x$ . They dominate everything else that appears, which is just this little  $1$ . So that's going to give you a ratio of  $1$ . If you don't trust that analysis, you can apply l'Hopital's rule again, and l'Hopital's rule will tell you exactly the equivalent thing. When you take a derivative, you'll be left with  $3e$  to the  $3x$  over  $3e$  to the  $3x$ , which is just  $1$ . So in either case, this part goes to  $1$ . So  $3/2$  times it goes right to  $3/2$ .

All right, so there you have it. Four limits computed with l'Hopital's rule. Well one of them was computed without l'Hopital's rule. Remember, you always have to check that you actually have

an indeterminate form before you apply l'Hopital's rule. And that's all I have to say about that.  
So I'll end there.