Finding the Radius of Convergence

Use the ratio test to find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Solution

As Christine explained in recitation, to find the radius of convergence of a series $\sum_{n=n_0}^{\infty} c_n x^n$ we apply the ratio test to find $L = \lim_{n\to\infty} \left| \frac{c_{n+1}x^{n+1}}{c_n x^n} \right|$. The value of x for which $L = 1$ is the radius of convergence of the power series.

In this case,

$$\begin{aligned} \frac{c_{n+1}x^{n+1}}{c_n x^n} &= \frac{x^{n+1}/(n+1)}{x^n/n} \\ &= x \cdot \frac{n}{n+1}. \end{aligned}$$

Taking the limit of this as n goes to infinity, we get:

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{c_{n+1}x^{n+1}}{c_n x^n} \right| \\ &= \lim_{n\to\infty} \left| x \cdot \frac{n}{n+1} \right| \\ &= \lim_{n\to\infty} \left| x \cdot \left(1 - \frac{1}{n+1} \right) \right| \\ L &= |x|. \end{aligned}$$

When $|x| < 1$, $L < 1$ and the ratio test tells us that the series will converge. For $|x| > 1$, $L > 1$ and the series diverges. The radius of convergence is 1.

This gives us an idea of how close the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is to being convergent.