18. Div grad curl and all that

Theorem 18.1. *Let $A \subset \mathbb{R}^n$ be open and let $f\colon A \longrightarrow \mathbb{R}$ be a differentiable function.*

If $\vec{r}\colon I \longrightarrow A$ is a flow line for $\nabla f\colon A \longrightarrow \mathbb{R}^n$, then the function $f \circ \vec{r}\colon I \longrightarrow \mathbb{R}$ is increasing.

Proof. By the chain rule,

$$\begin{aligned}\frac{d(f \circ \vec{r})}{dt}(t) &= \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \\ &= \vec{r}'(t) \cdot \vec{r}'(t) \geq 0.\end{aligned}$$

□

Corollary 18.2. *A closed parametrised curve is never the flow line of a conservative vector field.*

Once again, note that (18.2) is mainly a negative result:

Example 18.3.

$$\vec{F}\colon \mathbb{R}^2 - \{(0,0)\} \longrightarrow \mathbb{R}^2 \qquad \text{given by} \qquad \vec{F}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right),$$

is not a conservative vector field as it has flow lines which are circles.

Definition 18.4. *The* ***del operator*** *is the formal symbol*

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}.$$

Note that one can formally define the gradient of a function

$$\operatorname{grad} f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3,$$

by the formal rule

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}.$$

Using the operator del we can define two other operations, this time on vector fields:

Definition 18.5. *Let $A \subset \mathbb{R}^3$ be an open subset and let $\vec{F}\colon A \longrightarrow \mathbb{R}^3$ be a vector field.*

The ***divergence*** *of $\vec{F}$ is the scalar function,*

$$\operatorname{div} \vec{F}\colon A \longrightarrow \mathbb{R},$$

which is defined by the rule

$$\operatorname{div} \vec{F}(x,y,z) = \nabla \cdot \vec{F}(x,y,z) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}.$$

The ***curl*** *of* $\vec{F}$ *is the vector field*

$$\operatorname{curl}\vec{F}\colon A \longrightarrow \mathbb{R}^3,$$

which is defined by the rule

$$\begin{aligned}
\operatorname{curl}\vec{F}(x,x,z) &= \nabla \times \vec{F}(x,y,z) \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}.
\end{aligned}$$

Note that the del operator makes sense for any n, not just $n = 3$. So we can define the gradient and the divergence in all dimensions. However curl only makes sense when $n = 3$.

Definition 18.6. *The vector field* $\vec{F}\colon A \longrightarrow \mathbb{R}^3$ *is called* ***rotation free*** *if the curl is zero,* $\operatorname{curl}\vec{F} = \vec{0}$*, and it is called* ***incompressible*** *if the divergence is zero,* $\operatorname{div}\vec{F} = 0$.

Proposition 18.7. *Let* f *be a scalar field and* $\vec{F}$ *a vector field.*

(1) *If* f *is* $\mathcal{C}^2$*, then* $\operatorname{curl}(\operatorname{grad} f) = \vec{0}$*. Every conservative vector field is rotation free.*
(2) *If* $\vec{F}$ *is* $\mathcal{C}^2$*, then* $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$*. The curl of a vector field is incompressible.*

Proof. We compute;

$$\begin{aligned}
\operatorname{curl}(\operatorname{grad} f) &= \nabla \times (\nabla f) \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
&= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\hat{\imath} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\hat{\jmath} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\hat{k} \\
&= \vec{0}.
\end{aligned}$$

This gives (1).

$$\begin{aligned}
\operatorname{div}(\operatorname{curl}\vec{F}) &= \nabla\cdot(\nabla\times f)\\
&= \nabla\cdot\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}\\
&= \begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}\\
&= \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_1}{\partial y\partial z} + \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial z\partial y}\\
&= 0.
\end{aligned}$$

This is (2). □

Example 18.8. *The gravitational field*

$$\vec{F}(x,y,z) = \frac{cx}{(x^2+y^2+z^2)^{3/2}}\hat{\imath}+\frac{cy}{(x^2+y^2+z^2)^{3/2}}\hat{\jmath}+\frac{cz}{(x^2+y^2+z^2)^{3/2}}\hat{k},$$

is a gradient vector field, so that the gravitational field is rotation free. In fact if

$$f(x,y,z) = -\frac{c}{(x^2+y^2+z^2)^{1/2}},$$

then $\vec{F} = \operatorname{grad} f$*, so that*

$$\operatorname{curl}\vec{F} = \operatorname{curl}(\operatorname{grad} f) = \vec{0}.$$

Example 18.9. *A magnetic field* $\vec{B}$ *is always the curl of something,*

$$\vec{B} = \operatorname{curl}\vec{A},$$

where $\vec{A}$ *is a vector field. So*

$$\operatorname{div}(\vec{B}) = \operatorname{div}(\operatorname{curl}\vec{A}) = 0.$$

Therefore a magnetic field is always incompressible.

There is one other way to combine two del operators:

Definition 18.10. *The **Laplace operator** take a scalar field* $f\colon A \longrightarrow \mathbb{R}$ *and outputs another scalar field*

$$\nabla^2 f\colon A \longrightarrow \mathbb{R}.$$

It is defined by the rule

$$\nabla^2 f = \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x} + \frac{\partial^2 f}{\partial y} + \frac{\partial^2 f}{\partial z}.$$

A solution of the differential equation

$$\nabla^2 f = 0,$$

*is called a **harmonic function**.*

Example 18.11. *The function*

$$f(x, y, z) = -\frac{c}{(x^2 + y^2 + z^2)^{1/2}},$$

is harmonic.