26. Change of coordinates: II

Example 26.1. *Let D be the region bounded by the cardiod,*

$$r = 1 - \cos\theta.$$

If we multiply both sides by r, take $r\cos\theta$ over the other side, then we get

$$(x^2 + y^2 + x)^2 = x^2 + y^2.$$

We have

$$\begin{aligned}
\operatorname{area}(D) &= \iint_D \mathrm{d}x\,\mathrm{d}y \\
&= \iint_{D^*} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \left(\int_0^{1-\cos\theta} r\,\mathrm{d}r \right) \mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \left[\frac{r^2}{2} \right]_0^{1-\cos\theta} \mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \frac{(1-\cos\theta)^2}{2}\,\mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \frac{1}{2} - \cos\theta + \frac{\cos^2\theta}{2}\,\mathrm{d}\theta \\
&= \left[\frac{\theta}{2} - \sin\theta \right]_{-\pi}^{\pi} \frac{\pi}{2} \\
&= \frac{\pi}{2}.
\end{aligned}$$

In $\mathbb{R}^3$, we can either use cylindrical or spherical coordinates, instead of Cartesian coordinates.

Let's first do the case of cylindrical coordinates. Recall that

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z.
\end{aligned}$$

So the Jacobian is given by

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)}(r,\theta,z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

So,

$$\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{W^*} f(r,\theta,z)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.$$

Example 26.2. *Consider a cone of height b and base radius a. Put the vertex of the cone at the point $(0,0,b)$, so that the base of the cone is the circle of radius a, centred at the origin, in the xy-plane. Note that at height z, we have a circle of radius*

$$a\left(1-\frac{z}{b}\right).$$

$$\begin{aligned}
\mathrm{vol}(W) &= \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \iiint_{W^*} r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
&= \int_0^b \left(\int_0^{2\pi}\left(\int_0^{a(1-z/b} r\,\mathrm{d}r\right)\mathrm{d}\theta\right)\mathrm{d}z \\
&= \frac{1}{2}\int_0^b \left(\int_0^{2\pi} \left[r^2\right]_0^{a(1-z/b}\,\mathrm{d}\theta\right)\mathrm{d}z \\
&= \frac{1}{2}\int_0^b \left(\int_0^{2\pi} a^2\left(1-\frac{z}{b}\right)^2\mathrm{d}\theta\right)\mathrm{d}z \\
&= \pi a^2\int_{-a}^{a}\left(1-\frac{z}{b}\right)^2\mathrm{d}z \\
&= -\pi a^2 b\int_1^0 u^2\,\mathrm{d}u \\
&= \pi a^2 b\int_0^1 u^2\,\mathrm{d}u \\
&= \frac{\pi a^2 b}{3}.
\end{aligned}$$

Example 26.3. *Consider a ball of radius a. Put the centre of the ball at the point $(0,0,0)$. Note that*

$$x^2+y^2+z^2=a^2,$$

translates to the equation

$$r^2+z^2=a^2,$$

so that

$$r=\sqrt{a^2-z^2}.$$

$$
\begin{aligned}
\operatorname{vol}(W) &= \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \iint_{W^*} r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
&= \int_{-a}^{a}\left(\int_0^{2\pi}\left(\int_0^{\sqrt{a^2-z^2}} r\,\mathrm{d}r\right)\mathrm{d}\theta\right)\mathrm{d}z \\
&= \frac{1}{2}\int_{-a}^{a}\left(\int_0^{2\pi}\left[r^2\right]_0^{\sqrt{a^2-z^2}}\,\mathrm{d}\theta\right)\mathrm{d}z \\
&= \frac{1}{2}\int_{-a}^{a}\left(\int_0^{2\pi} a^2 - z^2\,\mathrm{d}\theta\right)\mathrm{d}z \\
&= \pi\int_{-a}^{a} a^2 - z^2\,\mathrm{d}z \\
&= \pi\left[a^2 z - \frac{z^3}{3}\right]_{-a}^{a} \\
&= \frac{4\pi a^3}{3}.
\end{aligned}
$$

Now consider using spherical coordinates. Recall that

$$
\begin{aligned}
x &= \rho\sin\phi\cos\theta \\
y &= \rho\sin\phi\sin\theta \\
z &= \rho\cos\phi.
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}(\rho,\phi,\theta) &= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi = \rho^2\sin\phi.
\end{aligned}
$$

Notice that this is greater than zero, if $0 < \phi < \pi$. So,

$$
\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{W^*} f(\rho,\phi,\theta)\rho^2\sin\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.
$$

Example 26.4. *Consider a ball of radius a. Put the centre of the ball at the point $(0, 0, 0)$.*

$$
\begin{aligned}
\mathrm{vol}(W) &= \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \iiint_{W^*} \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \left(\int_0^{\pi} \left(\int_0^{a} \rho^2 \sin\phi\,\mathrm{d}\rho \right) \mathrm{d}\phi \right) \mathrm{d}\theta \\
&= \int_0^{2\pi} \left(\int_0^{\pi} \sin\phi \left[\frac{\rho^3}{3} \right]_0^a \mathrm{d}\phi \right) \mathrm{d}\theta \\
&= \int_0^{2\pi} \left(\int_0^{\pi} \sin\phi \frac{a^3}{3}\,\mathrm{d}\phi \right) \mathrm{d}\theta \\
&= \frac{a^3}{3} \int_0^{2\pi} [-\cos\phi]_0^{\pi}\,\mathrm{d}\theta \\
&= \frac{2a^3}{3} \int_0^{2\pi} \mathrm{d}\theta \\
&= \frac{4\pi a^3}{3}.
\end{aligned}
$$