

28. MANIFOLDS WITH BOUNDARY

**Definition 28.1.** *Upper half space* is the set

$$\mathbb{H}^m = \{ (x_1, x_2, \dots, x_m) \mid x_m \geq 0 \} \subset \mathbb{R}^m.$$

The **boundary** of  $\mathbb{H}^m$ , is

$$\partial\mathbb{H}^m = \{ (x_1, x_2, \dots, x_m) \mid x_m = 0 \} \subset \mathbb{M}^m.$$

**Definition 28.2.** A subset  $M \subset \mathbb{R}^k$  is a **smooth  $m$ -manifold with boundary** if for every  $\vec{a} \in M$  there is an open subset  $W \subset \mathbb{R}^k$  and an open subset  $U \subset \mathbb{R}^m$ , and a diffeomorphism

$$\vec{g}: \mathbb{H}^m \cap U \longrightarrow M \cap W.$$

The **boundary** of  $M$  is the set of points  $\vec{a}$  which map to a point of the boundary of  $\mathbb{H}^m$ .

**Example 28.3.** The solid ellipse,

$$M = \{ (x, y) \in \mathbb{R}^2 \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1 \},$$

is a 2-manifold with boundary.

Let

$$U_1 = \{ (u, v) \mid 0 < u < 2\pi, -1 < v < 1 \}$$

and

$$W_1 = \mathbb{R}^2 - \{ (x, 0) \in \mathbb{R}^2 \mid x \geq 0 \}.$$

Define a function

$$\vec{g}_1: U_1 \longrightarrow W_1,$$

by the rule

$$\vec{g}_1(u, v) = (a(1-v)\cos u, b(1-v)\sin u).$$

Similarly, let

$$U_2 = \{ (u, v) \mid -\pi < u < \pi, -1 < v < 1 \}$$

and

$$W_2 = \mathbb{R}^2 - \{ (x, 0) \in \mathbb{R}^2 \mid x \leq 0 \}.$$

Define a function

$$\vec{g}_2: U_2 \longrightarrow W_2,$$

by the rule

$$\vec{g}_2(u, v) = (a(1-v)\cos u, b(1-v)\sin u).$$

Finally, let

$$U_3 = \{ (u, v) \mid u^2 + (v-b)^2 < b \}$$

and

$$W_3 = \{ (x, y) \mid x^2 + y^2 < b \}.$$

Define a function

$$\vec{g}_3: U_3 \longrightarrow W_3,$$

by the rule

$$\vec{g}_3(u, v) = (u, v - b).$$

Let

$$\vec{F}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2,$$

be the function

$$\vec{F}(x, y) = (-y, x).$$

Then

$$\begin{aligned} \int_{\partial M} \vec{F} \cdot d\vec{s} &= \int_0^{2\pi} (-b \sin t, a \cos t) \cdot (a \cos t, b \sin t) dt \\ &= \int_0^{2\pi} ab dt \\ &= 2\pi ab. \end{aligned}$$

On the other hand,

$$\text{curl } \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 - (-1) = 2,$$

and so

$$\iint_M \text{curl } \vec{F} dx dy = 2\pi ab.$$

In fact this is not a coincidence:

**Theorem 28.4** (Green's Theorem). *Let  $M \subset \mathbb{R}^2$  be a smooth 2-manifold with boundary, and let  $\vec{F}: M \longrightarrow \mathbb{R}^2$  be a smooth vector field such that*

$$\{ (x, y) \in M \mid \vec{F}(x, y) \neq \vec{0} \} \subset \mathbb{R}^2,$$

*is a bounded subset.*

*Then*

$$\iint_M \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy = \int_{\partial M} \vec{F} \cdot d\vec{s}.$$

*Here  $\partial M$  is oriented, so that  $M$  is on the left as we go around  $\partial M$  (in the positive direction).*

*Proof.* In the first step we assume that

$$M = \mathbb{H}^2 = \{ (u, v) \mid v \geq 0 \} \subset \mathbb{R}^2.$$

By assumption we may find  $a$  and  $b$  such that  $\vec{F} = \vec{0}$  outside the box

$$[-a/2, a/2] \times [0, b/2] \subset \mathbb{H}^2.$$

So  $\vec{F}(u, v) = \vec{0}$  if  $u = \pm a$  or  $v = b$ . Let's calculate the LHS,

$$\begin{aligned} \iint_{\mathbb{H}^2} \left( \frac{\partial F_2}{\partial u} - \frac{\partial F_1}{\partial v} \right) du dv &= \iint_{\mathbb{H}^2} \frac{\partial F_2}{\partial u} du dv - \iint_{\mathbb{H}^2} \frac{\partial F_1}{\partial v} du dv \\ &= \int_0^b \int_{-a}^a \frac{\partial F_2}{\partial u} du dv - \int_{-a}^a \int_0^b \frac{\partial F_1}{\partial v} dv du \\ &= \int_0^b F_2(a, v) - F_2(-a, v) dv - \int_{-a}^a F_1(u, b) - F_1(u, 0) du \\ &= \int_{-a}^a F_1(u, 0) du. \end{aligned}$$

Okay, now let's parametrise the boundary of the upper half plane,

$$\vec{x}: \mathbb{R} \longrightarrow \partial\mathbb{H}^2,$$

by the rule

$$\vec{x}(u) = (u, 0).$$

Then

$$\vec{x}'(u) = \hat{i}.$$

Let's calculate the RHS,

$$\begin{aligned} \int_{\partial\mathbb{H}^2} \vec{F} \cdot d\vec{s} &= \int_{-a}^a \vec{F}(\vec{x}(u)) \cdot \vec{x}'(u) du \\ &= \int_{-a}^a \vec{F}(u, 0) \cdot \hat{i} du \\ &= \int_{-a}^a F_1(u, 0) du. \end{aligned}$$

So the result holds if  $M = \mathbb{H}^2$ . This completes the first step.

In the second step, we suppose that there is a diffeomorphism

$$\vec{g}: \mathbb{H}^2 \cap U \longrightarrow M \cap W,$$

such that for some positive real numbers  $a$  and  $b$ , we have

- (1)  $[-a, a] \times [0, b] \subset \mathbb{H}^2 \cap U$ ,
- (2)  $\vec{F} = \vec{0}$  outside  $\vec{g}([-a/2, a/2] \times [0, b])$ , and
- (3)  $\det D\vec{g}(u, v) > 0$  for every  $(u, v) \in \mathbb{H}^2 \cap U$ .

In this case, parametrise  $\partial M \cap W$  as follows; define

$$\vec{s}: (-a, a) \longrightarrow \partial M \cap W,$$

by the rule

$$\vec{s}(u) = \vec{g}(\vec{x}(u)) = \vec{g}(u, 0).$$

Note that this is compatible with the orientation, as we are assuming that the Jacobian of  $g$  is positive.

$$\begin{aligned} \int_{\partial M} \vec{F} \cdot d\vec{s} &= \int_{-a}^a \vec{F}(\vec{s}(u)) \cdot \vec{s}'(u) du \\ &= \int_{-a}^a \vec{F}(\vec{g}(\vec{x}(u))) D\vec{g}(\vec{x}(u)) \cdot \vec{x}'(u) du \\ &= \int_{\partial \mathbb{H}^2} \vec{G} \cdot d\vec{s}, \end{aligned}$$

where

$$\vec{G}: \mathbb{H}^2 \longrightarrow \mathbb{R}^2,$$

is defined by the rule

$$\vec{G}(u, v) = \begin{cases} \vec{F}(\vec{g}(u, v)) D\vec{g}(u, v) & \text{if } (u, v) \in U \\ \vec{0} & \text{otherwise.} \end{cases}$$

Now we compute,

$$\begin{aligned} \frac{\partial G_2}{\partial u} - \frac{\partial G_1}{\partial v} &= \frac{\partial}{\partial u} \left( F_1 \frac{\partial x}{\partial v} + F_2 \frac{\partial y}{\partial v} \right) - \frac{\partial}{\partial v} \left( F_1 \frac{\partial x}{\partial u} + F_2 \frac{\partial y}{\partial u} \right) \\ &= \left( \frac{\partial F_1}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial F_1}{\partial y} \frac{\partial y}{\partial u} \right) \frac{\partial x}{\partial v} + F_1 \frac{\partial^2 x}{\partial u \partial v} \\ &\quad + \left( \frac{\partial F_2}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial F_2}{\partial y} \frac{\partial y}{\partial u} \right) \frac{\partial y}{\partial v} + F_2 \frac{\partial^2 y}{\partial u \partial v} \\ &\quad - \left( \frac{\partial F_1}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial F_1}{\partial y} \frac{\partial y}{\partial v} \right) \frac{\partial x}{\partial u} - F_1 \frac{\partial^2 x}{\partial v \partial u} \\ &\quad - \left( \frac{\partial F_2}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial F_2}{\partial y} \frac{\partial y}{\partial v} \right) \frac{\partial y}{\partial u} - F_2 \frac{\partial^2 y}{\partial v \partial u} \\ &= \frac{\partial F_2}{\partial x} \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) - \frac{\partial F_1}{\partial y} \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) \\ &= \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \frac{\partial(x, y)}{\partial(u, v)}. \end{aligned}$$

Using this, we get

$$\begin{aligned}
\iint_{\mathbb{H}^2} \left( \frac{\partial G_2}{\partial u} - \frac{\partial G_1}{\partial v} \right) du dv &= \iint_{\mathbb{H}^2 \cap U} \left( \frac{\partial G_2}{\partial u} - \frac{\partial G_1}{\partial v} \right) du dv \\
&= \iint_{\mathbb{H}^2 \cap U} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \frac{\partial(x, y)}{\partial(u, v)} du dv \\
&= \iint_{M \cap W} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy \\
&= \iint_M \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy.
\end{aligned}$$

Putting all of this together, we have

$$\begin{aligned}
\iint_M \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy &= \iint_{\mathbb{H}^2} \left( \frac{\partial G_2}{\partial u} - \frac{\partial G_1}{\partial v} \right) du dv \\
&= \int_{\partial \mathbb{H}^2} \vec{G} \cdot d\vec{s} \\
&= \int_{\partial M} \vec{F} \cdot d\vec{s}.
\end{aligned}$$

This completes step 2.

We now turn to the third and final step. To complete the proof, we need to invoke the existence of partitions of unity. Starting with  $\vec{F}$ , I claim that there are vector finitely many fields  $F_1, F_2, \dots, F_k$ , each of which satisfy the hypotheses of step 2, such that

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_k = \sum_{i=1}^k \vec{F}_i.$$

Indeed, start with a partition of unity,

$$1 = \sum_{i=1}^m \rho_i,$$

multiply both sides by  $\vec{F}$ , to get

$$\vec{F} = \sum_{i=1}^m \rho_i \vec{F} = \sum_{i=1}^m \vec{F}_i.$$

Granted this, Green's Theorem follows very easily,

$$\begin{aligned} \iint_M \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy &= \sum_{i=1}^k \iint_M \left( \frac{\partial F_{i,2}}{\partial x} - \frac{\partial F_{i,1}}{\partial y} \right) dx dy \\ &= \sum_{i=1}^k \int_{\partial M} \vec{F}_i \cdot d\vec{s}. \\ &= \int_{\partial M} \vec{F} \cdot d\vec{s}. \quad \square \end{aligned}$$

**Lemma 28.5.** *Let  $K \subset \mathbb{R}^n$ . Suppose that  $K$  is contained in the union of closed balls  $B_1, B_2, \dots, B_m$ , such that any point of  $K$  belongs to the interior of at least one of  $B_1, B_2, \dots, B_m$ .*

*Then we may find smooth functions  $\rho_1, \rho_2, \dots, \rho_m$  such that  $\rho_i$  is zero outside  $B_i$  and*

$$1 = \sum_{i=1}^m \rho_i.$$

*Proof.* We prove the case  $n = 2$ . The general case is similar, only notationally more involved. First observe that it is enough to find smooth functions  $\sigma_1, \sigma_2, \dots, \sigma_m$ , such that  $\sigma_i$  is zero outside  $B_i$  and such that

$$\sigma = \sum_{i=1}^m \sigma_i,$$

does not vanish at any point of  $K$ . Indeed, if we let

$$\rho_i = \frac{\sigma_i}{\sigma},$$

then  $\rho_i$  is smooth, it vanishes outside  $B_i$  and dividing both sides of the equation above by  $\sigma$ , we have

$$1 = \sum_{i=1}^m \rho_i.$$

In fact it suffices to find functions  $\sigma_1, \sigma_2, \dots, \sigma_m$ , such that  $\sigma_i$  vanishes outside  $B_i$  and which is non-zero on the interior of  $B_i$  (replacing  $\sigma_i$  by  $\sigma_i^2$ , so that  $\sigma_i^2$  is positive on the interior of  $B_i$ , we get rid of the annoying possibility that the sum is zero because of cancelling). It is enough to do this for one solid circle  $B_i$  and we might as well assume that  $B = B_m = B_1$  is the solid unit circle. Using polar coordinates, we want a function of one variable  $r$  which is zero outside  $[0, 1]$  and which is non-zero on  $(0, 1)$ , so we are now down to a one variable question.

At this point we realise we want a smooth function,

$$f: \mathbb{R} \longrightarrow \mathbb{R},$$

all of whose derivatives are zero at 0 and yet the function  $f$  is not the zero function. Such a function is given by

$$f(x) = e^{-1/x^2}. \quad \square$$

MIT OpenCourseWare  
<http://ocw.mit.edu>

18.022 Calculus of Several Variables  
Fall 2010

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.