

29. CONSERVATIVE VECTOR FIELDS REVISITED

Let $U \subset \mathbb{R}^2$ be an open subset. Given a smooth function $f: U \rightarrow \mathbb{R}$ we get a smooth vector field by taking $\vec{F} = \text{grad } f$. Given a smooth vector field $\vec{F}: U \rightarrow \mathbb{R}^2$ we get a function by taking $f = \text{curl } \vec{F}$.

Suppose that $M \subset U$ is a smooth 2-manifold with boundary. If we start with \vec{F} , then we have

$$\iint_M \text{curl } \vec{F} \, dA = \int_{\partial M} \vec{F} \cdot d\vec{s}.$$

Suppose we start with f , and let C be a smooth oriented curve. Pick a parametrisation,

$$\vec{x}: [a, b] \rightarrow U,$$

such that $\vec{x}(a) = P$ and $\vec{x}(b) = Q$. Then we have

$$\begin{aligned} \int_C \text{grad } f \cdot d\vec{s} &= \int_a^b \text{grad } f(\vec{x}(t)) \cdot \vec{x}'(t) \, dt \\ &= \int_a^b \frac{d}{dt} f(\vec{x}(t)) \, dt \\ &= f(\vec{x}(b)) - f(\vec{x}(a)) \\ &= f(Q) - f(P). \end{aligned}$$

Definition 29.1. We say that $X \subset \mathbb{R}^n$ is **star-shaped** with respect to $P \in X$, if given any point $Q \in X$ then the point

$$P + t\overrightarrow{PQ} \in X,$$

for every $t \in [0, 1]$.

In other words, the line segment connecting P to Q belongs to X .

Theorem 29.2. Let $U \subset \mathbb{R}^2$ be an open and star-shaped let $\vec{F}: U \rightarrow \mathbb{R}^2$ be a smooth vector field.

The following are equivalent:

- (1) $\text{curl } \vec{F} = 0$.
- (2) $\vec{F} = \text{grad } f$.

Proof. (2) implies (1) is easy.

We check (1) implies (2). Suppose that U is star-shaped with respect to $P = (x_0, y_0)$. Parametrise the line L from P to $Q = (x, y)$ as follows

$$P + t\overrightarrow{PQ} = (x_0 + t(x - x_0), y_0 + t(y - y_0)) = P_t, \quad \text{for } 0 \leq t \leq 1.$$

Define

$$\begin{aligned} f(x, y) &= \int_L \vec{F} \cdot d\vec{s} \\ &= \int_0^1 xF_1(P_t) + yF_2(P_t) dt. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial f}{\partial x} &= \int_0^1 \frac{\partial}{\partial x} (xF_1(x_0 + t(x - x_0), y_0 + t(y - y_0)) + yF_2(x_0 + t(x - x_0), y_0 + t(y - y_0))) dt \\ &= \int_0^1 F_1(P_t) + tx \frac{\partial F_1}{\partial x}(P_t) + ty \frac{\partial F_2}{\partial x}(P_t) dt. \end{aligned}$$

On the other hand,

$$\frac{\partial}{\partial t} (tF_1(x_0 + t(x - x_0), y_0 + t(y - y_0))) = F_1(P_t) + tx \frac{\partial F_1}{\partial t}(P_t) + ty \frac{\partial F_2}{\partial y}(P_t).$$

Since $\text{curl } \vec{F} = 0$, we have

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x},$$

and so

$$\frac{\partial f}{\partial x} = \int_0^1 \frac{\partial F_1}{\partial t}(P_t) dt = F_1(x, y).$$

Similarly

$$\frac{\partial f}{\partial y} = F_2(x, y).$$

It follows that $\vec{F} = \text{grad } f$. □

Definition 29.3. Let $\vec{F}: U \rightarrow \mathbb{R}^2$ be a vector field. Define another vector field by the rule

$$*F: U \rightarrow \mathbb{R}^2 \quad \text{where} \quad *F = (-F_2, F_1).$$

Theorem 29.4 (Divergence theorem in the plane). Suppose that $M \subset \mathbb{R}^2$ is a smooth 2-manifold with boundary ∂M .

If $\vec{F}: U \rightarrow \mathbb{R}^2$ is a smooth vector field, then

$$\iint_M \text{div } \vec{F} dA = \int_{\partial M} \vec{F} \cdot \hat{n} ds,$$

where \hat{n} is the unit normal vector of the smooth oriented curve $C = \partial M$ which points out of M .

Proof. Note that

$$\operatorname{curl}(*\vec{F}) = \operatorname{div} \vec{F},$$

and

$$*F \cdot d\vec{s} = (\vec{F} \cdot \hat{n})ds,$$

and so the result follows from Green's theorem applied to $*\vec{F}$. \square

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