30. Surface integrals

Suppose we are given a smooth 2-manifold $M \subset \mathbb{R}^3$. Let

$$\vec{g}\colon U \longrightarrow M \cap W,$$

be a diffeomorphism, where $U \subset \mathbb{R}^2$, with coordinates s and t.

We can define two tangent vectors, which span the tangent plane to M at $P = \vec{g}(s_0, t_0)$:

$$\vec{T}_s(s_0, t_0) = \frac{\partial \vec{g}}{\partial s}(s_0, t_0)$$
$$\vec{T}_t(s_0, t_0) = \frac{\partial \vec{g}}{\partial t}(s_0, t_0).$$

We get an element of area on M,

$$\mathrm{d}S = \|\vec{T}_s \times \vec{T}_t\| \, \mathrm{d}s \, \mathrm{d}t.$$

Using this we can define the area of $M \cap W$ to be

$$\operatorname{area}(M \cap W) = \iint_{M \cap W} \mathrm{d}S = \iint_U \|\vec{T}_s \times \vec{T}_t\| \, \mathrm{d}s \, \mathrm{d}t.$$

Example 30.1. *We can parametrise the torus,*

$$M = \{\, (x, y, z) \mid (a - \sqrt{x^2 + y^2})^2 + z^2 = b^2 \,\},$$

as follows. Let

$$U = (0, 2\pi) \times (0, 2\pi),$$

and

$$W = \mathbb{R}^3 \setminus \{\, (x, y, z) \mid x \geq 0 \textit{ and } y = 0, \textit{ or } x^2 + y^2 \geq a^2 \textit{ and } z = 0 \,\}.$$

Let

$$\vec{g}\colon U \longrightarrow M \cap W,$$

be the function

$$\vec{g}(s, t) = ((a + b\cos t)\cos s, (a + b\cos t)\sin s, b\sin t).$$

Let's calculate the tangent vectors,

$$\vec{T}_s = \frac{\partial \vec{g}}{\partial s} = (-(a + b\cos t)\sin s, (a + b\cos t)\cos s, 0),$$
$$\vec{T}_t = \frac{\partial \vec{g}}{\partial t} = (-b\sin t\cos s, -b\sin t\sin s, b\cos t).$$

So

$$\begin{aligned} \vec{T}_s \times \vec{T}_t &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -(a + b\cos t)\sin s & (a + b\cos t)\cos s & 0 \\ -b\sin t\cos s & -b\sin t\sin s & b\cos t \end{vmatrix} \\ &= (a + b\cos t)b\cos s\cos t\hat{\imath} + (a + b\cos t)b\sin s\cos t\hat{\jmath} + (a + b\cos t)b\sin t\hat{k}. \end{aligned}$$

Therefore,

$$\begin{aligned}\|\vec{T}_s \times \vec{T}_t\| &= (a + b\cos t)b(\cos^2 s\cos^2 t + \sin^2 s\cos^2 t + \sin^2 t)^{1/2} \\ &= (a + b\cos t)b.\end{aligned}$$

As $a \geq b$, note that $(a + b\cos t)b > 0$. Hence

$$\begin{aligned}\operatorname{area}(M) &= \operatorname{area}(M \cap W) \\ &= \iint_{M\cap W} \mathrm{d}S \\ &= \iint_U \|\vec{T}_s \times \vec{T}_t\| \,\mathrm{d}s\,\mathrm{d}t \\ &= \int_0^{2\pi}\int_0^{2\pi} (a + b\cos t)b \,\mathrm{d}s\,\mathrm{d}t \\ &= 2\pi b\int_0^{2\pi} (a + b\cos t)\,\mathrm{d}t \\ &= 4\pi^2 ab.\end{aligned}$$

Notice that this is the surface area of a cylinder of radius b *and height* $2\pi a$, *as expected.*

Example 30.2. *We can parametrise the sphere,*

$$M = \{\, (x, y, z) \mid x^2 + y^2 + z^2 = a^2 \,\},$$

as follows. Let

$$U = (0, \pi) \times (0, 2\pi),$$

and

$$W = \mathbb{R}^3 \setminus \{\, (x, y, z) \mid x \geq 0 \text{ and } y = 0 \,\}.$$

Let

$$\vec{g} \colon U \longrightarrow M \cap W,$$

be the function

$$\vec{g}(\phi, \theta) = (a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi).$$

Let's calculate the tangent vectors,

$$\begin{aligned}\vec{T}_\phi &= \frac{\partial \vec{g}}{\partial \phi} = (a\cos\phi\cos\theta, a\cos\phi\sin\theta, -a\sin\phi), \\ \vec{T}_\theta &= \frac{\partial \vec{g}}{\partial \theta} = (-a\sin\phi\sin\theta, a\sin\phi\cos\theta, 0).\end{aligned}$$

So

$$\begin{aligned}\vec{T}_\phi \times \vec{T}_\theta &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix} \\ &= a^2\sin^2\phi\cos\theta\hat{\imath} + a^2\sin^2\phi\sin\theta\hat{\jmath} + a^2\cos\phi\sin\phi\hat{k}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\vec{T}_\phi \times \vec{T}_\theta\| &= a^2\sin\phi(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi)^{1/2} \\ &= a^2\sin\phi.\end{aligned}$$

As $0 < \phi < \pi$, *note that* $a^2\sin\phi > 0$. *Hence*

$$\begin{aligned}\operatorname{area}(M) &= \operatorname{area}(M \cap W) \\ &= \iint_{M\cap W} \mathrm{d}S \\ &= \iint_U \|\vec{T}_\phi \times \vec{T}_\theta\| \,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= 2a^2\int_0^{2\pi}\mathrm{d}t \\ &= 4\pi a^2.\end{aligned}$$

Notice that this is the surface area of a sphere of radius a.

Let's now suppose that there are two different ways to parametrise the same piece $M \cap W$ of the manifold M:

$$\vec{g}\colon U \longrightarrow M \cap W \qquad \text{and} \qquad \vec{h}\colon V \longrightarrow M \cap W.$$

Let use (u, v) coordinates for $U \subset \mathbb{R}^2$ and (s, t) coordinates for $V \subset \mathbb{R}^2$. Then

$$\vec{f} = (\vec{h})^{-1} \circ \vec{g}\colon U \longrightarrow V,$$

is a diffeomorphism. Note that $\vec{g} = \vec{h} \circ \vec{f}$. We then have

$$\begin{aligned}\frac{\partial\vec{g}}{\partial u}(u, v) &= \frac{\partial(\vec{h} \circ \vec{f})}{\partial u}(u, v) \\ &= \frac{\partial\vec{h}}{\partial s}(s, t)\frac{\partial s}{\partial u}(u, v) + \frac{\partial\vec{h}}{\partial t}(s, t)\frac{\partial t}{\partial u}(u, v).\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial\vec{g}}{\partial v}(u,v) &= \frac{\partial(\vec{h}\circ\vec{f})}{\partial v}(u,v)\\ &= \frac{\partial\vec{h}}{\partial s}(s,t)\frac{\partial s}{\partial v}(u,v) + \frac{\partial\vec{h}}{\partial t}(s,t)\frac{\partial t}{\partial v}(u,v).\end{aligned}$$

$$\begin{aligned}\frac{\partial\vec{g}}{\partial u}\times\frac{\partial\vec{g}}{\partial v} &= \left(\frac{\partial\vec{h}}{\partial s}\frac{\partial s}{\partial u} + \frac{\partial\vec{h}}{\partial t}\frac{\partial t}{\partial u}\right)\times\left(\frac{\partial\vec{h}}{\partial s}\frac{\partial s}{\partial v} + \frac{\partial\vec{h}}{\partial t}\frac{\partial t}{\partial v}\right)\\ &= \frac{\partial\vec{h}}{\partial s}\times\frac{\partial\vec{h}}{\partial t}\left(\frac{\partial s}{\partial u}\frac{\partial t}{\partial v} - \frac{\partial s}{\partial v}\frac{\partial t}{\partial u}\right)\\ &= \frac{\partial\vec{h}}{\partial s}\times\frac{\partial\vec{h}}{\partial t}\frac{\partial(s,t)}{\partial(u,v)}.\end{aligned}$$

It follows that

$$\|\frac{\partial\vec{g}}{\partial u}\times\frac{\partial\vec{g}}{\partial v}\| = \|\frac{\partial\vec{h}}{\partial s}\times\frac{\partial\vec{h}}{\partial t}\||\frac{\partial(s,t)}{\partial(u,v)}|.$$

Hence

$$\begin{aligned}\iint_U \|\frac{\partial\vec{g}}{\partial u}\times\frac{\partial\vec{g}}{\partial v}\|\,\mathrm{d}u\,\mathrm{d}v &= \iint_U \|\frac{\partial\vec{h}}{\partial s}\times\frac{\partial\vec{h}}{\partial t}\||\frac{\partial(s,t)}{\partial(u,v)}|\,\mathrm{d}u\,\mathrm{d}v\\ &= \iint_V \|\frac{\partial\vec{h}}{\partial s}\times\frac{\partial\vec{h}}{\partial t}\|\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

Notice that the first term is precisely the integral we use to define the area of $M\cap W$. This formula then says that the area is independent of the choice of parametrisation.

18.022 Calculus of Several Variables
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