31. Flux

Let $S \subset \mathbb{R}^3$ be a smooth 2-manifold and let

$$\vec{g} \colon U \longrightarrow S \cap W,$$

be a diffeomorphism.

Definition 31.1. *Let $f \colon S \cap W \longrightarrow \mathbb{R}$ and $\vec{F} \colon S \cap W \longrightarrow \mathbb{R}^3$ be two functions, the first a scalar function and the second a vector field. We define*

$$\iint_{S\cap W} f \,\mathrm{d}S = \iint_{S\cap W} f(g(s,t)) \|\frac{\partial \vec{g}}{\partial s} \times \frac{\partial \vec{g}}{\partial t}\| \,\mathrm{d}s\,\mathrm{d}t$$
$$\iint_{S\cap W} \vec{F} \cdot \mathrm{d}\vec{S} = \iint_{S\cap W} \vec{F}(g(s,t)) \cdot \left(\frac{\partial \vec{g}}{\partial s} \times \frac{\partial \vec{g}}{\partial t}\right) \mathrm{d}s\,\mathrm{d}t.$$

The second integral is called the ***flux of $\vec{F}$ across S in the direction of***

$$\hat{n} = \frac{\frac{\partial \vec{g}}{\partial s} \times \frac{\partial \vec{g}}{\partial t}}{\|\frac{\partial \vec{g}}{\partial s} \times \frac{\partial \vec{g}}{\partial t}\|}.$$

Note that

$$\iint_{S\cap W} \vec{F} \cdot \mathrm{d}\vec{S} = \iint_{S\cap W} (\vec{F} \cdot \hat{n}) \,\mathrm{d}\vec{S}.$$

Note also that one can define the line integral of f and $\vec{F}$ over the whole of S using partitions of unity.

Example 31.2. *Find the flux of the vector field given by*

$$\vec{F}(x,y,z) = y\hat{\imath} + z\hat{\jmath} + x\hat{k},$$

through the triangle S with vertices

$$P_0 = (1,2,-1) \qquad P_1 = (2,1,1) \qquad \textit{and} \qquad P_2 = (3,-1,2),$$

in the direction of

$$\hat{n} = \frac{\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}}{\|\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}\|}.$$

First we parametrise S,

$$\vec{g} \colon U \longrightarrow S \cap W,$$

where

$$g(s,t) = \overrightarrow{OP_0} + s\overrightarrow{P_0P_1} + t\overrightarrow{P_0P_2} = (1+s+2t, 2-s-3t, -1+2s+3t),$$

and

$$U = \{\, (s,t) \in \mathbb{R}^2 \mid 0 < s < 1, 0 < t < 1-s \,\},$$

and W is the whole of $\mathbb{R}^3$ minus the three lines P_0P_1, P_1P_2 and P_2P_0. Now

$$\frac{\partial \vec{g}}{\partial s} = \overrightarrow{P_0P_1} = (1,-1,2) \qquad \text{and} \qquad \frac{\partial \vec{g}}{\partial t} = \overrightarrow{P_0P_2} = (2,-3,3),$$

and so

$$\begin{aligned}\frac{\partial \vec{g}}{\partial s} \times \frac{\partial \vec{g}}{\partial t} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 2 \\ 2 & -3 & 3 \end{vmatrix} \\ &= 3\hat{\imath} + \hat{\jmath} - \hat{k}.\end{aligned}$$

Clearly, $\hat{n}$ and $\frac{\partial \vec{g}}{\partial s} \times \frac{\partial \vec{g}}{\partial t}$ have the same direction.

$$\begin{aligned}\iint_S \vec{F} \cdot \mathrm{d}\vec{S} &= \iint_{S \cap W} \vec{F} \cdot \mathrm{d}\vec{S} \\ &= \iint_U \vec{F}(\vec{g}(s,t)) \cdot \left(\frac{\partial \vec{g}}{\partial s} \times \frac{\partial \vec{g}}{\partial t} \right) \mathrm{d}s\, \mathrm{d}t \\ &= \int_0^1 \int_0^{1-s} (2-s-3t, -1+2s+3t, 1+s+2t) \cdot (3,1,-1)\, \mathrm{d}t\, \mathrm{d}s \\ &= \int_0^1 \int_0^{1-s} (6-3s-9t-1+2s+3t-1-s-2t)\, \mathrm{d}t\, \mathrm{d}s \\ &= \int_0^1 \int_0^{1-s} (4-2s-8t)\, \mathrm{d}t\, \mathrm{d}s \\ &= \int_0^1 \left[4t - 2st - 4t^2\right]_0^{1-s} \mathrm{d}s \\ &= \int_0^1 (4(1-s) - 2s(1-s) - 4(1-s)^2)\, \mathrm{d}s \\ &= \int_0^1 (2s - 2s^2)\, \mathrm{d}s \\ &= \left[s^2 - \frac{2s^3}{3} \right]_0^1 \\ &= \frac{1}{3}.\end{aligned}$$

Example 31.3. *Let S be the disk of radius 2 centred around the point $P = (1,1,-2)$ and orthogonal to the vector*

$$\hat{n} = (\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}).$$

Find the flux of the vector field given by

$$\vec{F}(x,y,z) = y\hat{\imath} + z\hat{\jmath} + x\hat{k},$$

through S in the direction of $\hat{n}$.

First we need to parametrise S. We want a right handed triple of unit vectors

$$(\hat{a}, \hat{b}, \hat{n})$$

which are pairwise orthogonal, so that they are an orthonormal basis. Let's take

$$\hat{a} = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}).$$

With this choice, it is clear that

$$\hat{a} \cdot \hat{n} = 0,$$

so that $\hat{a}$ is orthogonal to $\hat{n}$,

$$\begin{aligned} \hat{b} &= \hat{n} \times \hat{a} \\ &= \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{\jmath} \\ 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix} \\ &= -2/3\hat{\imath} + 1/3\hat{\jmath} + 2/3\hat{k}. \end{aligned}$$

This gives us a parametrisation,

$$\vec{g} \colon U \longrightarrow S \cap W,$$

given by

$$\begin{aligned} g(r,\theta) &= \overrightarrow{OP} + r\cos\theta\hat{a} + r\sin\theta\hat{b} \\ &= (1,1,-2) + (2r/3\cos\theta, 2r/3\cos\theta, r/3\cos\theta) + (-2r/3\sin\theta, r/3\sin\theta, 2r/3\sin\theta) \\ &= (1 + 2r/3\cos\theta - 2r/3\sin\theta, 1 + 2r/3\cos\theta + r/3\sin\theta, -2 + r/3\cos\theta + 2r/3\sin\theta), \end{aligned}$$

where

$$U = (0,2) \times (0, 2\pi),$$

and W is the whole of $\mathbb{R}^3$ minus the boundary of the disk.

Now

$$\frac{\partial \vec{g}}{\partial r} = \cos\theta\hat{a} + \sin\theta\hat{b} \qquad \text{and} \qquad \frac{\partial \vec{g}}{\partial \theta} = -r\sin\theta\hat{a} + r\cos\theta\hat{b},$$

and so

$$\begin{aligned} \frac{\partial \vec{g}}{\partial r} \times \frac{\partial \vec{g}}{\partial \theta} &= (\cos\theta\hat{a} + \sin\theta\hat{b}) \times (-r\sin\theta\hat{a} + r\cos\theta\hat{b}) \\ &= r(\cos^2\theta + \sin^2\theta)\hat{n} = r\hat{n}. \end{aligned}$$

Clearly this points in the direction of $\hat{n}$.

$$
\begin{aligned}
\iint_S \vec{F}\cdot \mathrm{d}\vec{S} &= \iint_{S\cap W} \vec{F}\cdot \mathrm{d}\vec{S} \\
&= \iint_U \vec{F}(\vec{g}(r,\theta))\cdot \left(\frac{\partial \vec{g}}{\partial r}\times \frac{\partial \vec{g}}{\partial \theta}\right) \mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^2 (1+2r/3\cos\theta + r/3\sin\theta, -2+r/3\cos\theta+2r/3\sin\theta, \\
&\quad 1+2r/3\cos\theta - 2r/3\sin\theta,)\cdot(r/3,-2r/3,2r/3)\,\mathrm{d}r\,\mathrm{d}\theta.
\end{aligned}
$$

Now when we expand the integrand, we will clearly get

$$\alpha+\beta\cos\theta+\gamma\sin\theta,$$

where α, β and γ are affine linear functions of r (that is, of the form $mr+b$). The integral of $\cos\theta$ and $\sin\theta$ over the range $[0,2\pi]$ is zero. Computing, we get

$$\alpha = r(1/3+4/3+2/3) = 7r/3,$$

so that α is a linear function of r. Therefore the integral reduces to

$$
\begin{aligned}
\int_0^{2\pi}\int_0^2 \frac{7r}{3}\,\mathrm{d}r\,\mathrm{d}\theta &= \int_0^{2\pi}\left[\frac{7r^2}{6}\right]_0^2 \mathrm{d}\theta \\
&= \int_0^{2\pi}\frac{14}{3}\,\mathrm{d}\theta \\
&= \frac{28\pi}{3}.
\end{aligned}
$$