

34. FORMS ON \mathbb{R}^n

Definition 34.1. A **basic 1-form** on \mathbb{R}^n is a formal symbol

$$dx_1, dx_2, \dots, dx_n.$$

A **general 1-form** on \mathbb{R}^n is any expression of the form

$$\omega = f_1 dx_1 + f_2 dx_2 + \dots + f_n dx_n,$$

where f_1, f_2, \dots, f_n are smooth functions.

Note that there are n basic 1-forms. If f is a smooth function, we get a 1-form using the formal rule,

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i.$$

Definition 34.2. A **basic 2-form** on \mathbb{R}^n is any formal symbol

$$dx_i \wedge dx_j,$$

where $1 \leq i < j \leq n$. A **general 2-form** on \mathbb{R}^n is any expression of the form

$$\omega = \sum_{i < j} f_{ij} dx_i \wedge dx_j$$

where f_1, f_2, \dots, f_n are smooth functions.

Note that there are

$$\binom{n}{2} = \frac{n(n-1)}{2},$$

basic 2-forms. 2-forms satisfy the following formal rule,

$$dx_i \wedge dx_j = -dx_j \wedge dx_i.$$

Given two 1-forms,

$$\omega = f_1 dx_1 + f_2 dx_2 + \dots + f_n dx_n \quad \text{and} \quad \eta = g_1 dx_1 + g_2 dx_2 + \dots + g_n dx_n,$$

we can multiply, using linearity and skew-commutativity, to get a 2-form

$$\omega \wedge \eta = \sum_{i < j} (f_i g_j - f_j g_i) dx_i \wedge dx_j.$$

Notice how, using skew-commutativity, we can write everything in terms of basic 2-forms. Note also that skew-commutativity forces

$$dx_i \wedge dx_i = 0.$$

More generally:

Definition 34.3. A **basic k -form** on \mathbb{R}^n is a formal symbol

$$dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k},$$

where $1 \leq i_1 < i_2 < \dots < i_k \leq n$. A **general k -form** on \mathbb{R}^n is any expression of the form

$$\omega = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} f_{i_1 i_2 \dots i_k} dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k},$$

where $f_{i_1 i_2 \dots i_k}$ are smooth functions.

Note that there are

$$\binom{n}{k},$$

basic n -forms.

Suppose we start with a general 1-form in \mathbb{R}^3 ,

$$\omega = f_1 dx + f_2 dy + f_3 dz.$$

We can formally define a 2-form $d\omega$ as follows:

$$\begin{aligned} d\omega &= d(f_1 dx + f_2 dy + f_3 dz) \\ &= df_1 \wedge dx + df_2 \wedge dy + df_3 \wedge dz \\ &= \left(\frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial y} dy + \frac{\partial f_1}{\partial z} dz \right) \wedge dx \\ &\quad + \left(\frac{\partial f_2}{\partial x} dx + \frac{\partial f_2}{\partial y} dy + \frac{\partial f_2}{\partial z} dz \right) \wedge dy \\ &\quad + \left(\frac{\partial f_3}{\partial x} dx + \frac{\partial f_3}{\partial y} dy + \frac{\partial f_3}{\partial z} dz \right) \wedge dz \\ &= \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx \wedge dy + \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) dx \wedge dz + \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) dy \wedge dz. \end{aligned}$$

Note also that if

$$\omega = f_{12} dx \wedge dy + f_{13} dx \wedge dz + f_{23} dy \wedge dz,$$

then we can formally define a 3-form $d\omega$ as follows:

$$\begin{aligned} d\omega &= d(f_{12} dx \wedge dy + f_{13} dx \wedge dz + f_{23} dy \wedge dz,) \\ &= \left(\frac{\partial f_{12}}{\partial z} - \frac{\partial f_{13}}{\partial y} + \frac{\partial f_{23}}{\partial x} \right) dx \wedge dy \wedge dz. \end{aligned}$$

The final interesting formal operation on k -forms is:

Definition 34.4. Let ω be a k -form on \mathbb{R}^n . The **Hodge star operator** assigns an $(n - k)$ -form $*\omega$ to ω , which is defined by the rule:

$$(*\omega) \wedge \omega = dx_1 \wedge dx_2 \wedge \dots \wedge dx_n.$$

Let's figure out what the Hodge star operator does in \mathbb{R}^3 . Suppose that

$$\omega = dx \wedge dy.$$

Then $\eta = *\omega$ is a 1-form such that

$$\eta \wedge \omega = dx \wedge dy \wedge dz.$$

It follows that $\eta = dz$. Similarly

$$*(dx \wedge dz) = -dy \quad \text{and} \quad *(dy \wedge dz) = dx.$$

Let

$$\omega = f_1 dx + f_2 dy + f_3 dz.$$

Putting all of this together, we see that

$$\begin{aligned} *d\omega &= *d(f_1 dx + f_2 dy + f_3 dz) \\ &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) dx - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) dy + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dz. \end{aligned}$$

Similarly

$$\begin{aligned} *d*\omega &= *d*(f_1 dx + f_2 dy + f_3 dz) \\ &= *d(f_1 dy \wedge dz - f_2 dx \wedge dz + f_3 dx \wedge dy) \\ &= * \left(\frac{\partial f_3}{\partial z} + \frac{\partial f_2}{\partial y} + \frac{\partial f_1}{\partial x} \right) dx \wedge dy \wedge dz \\ &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}. \end{aligned}$$

In other words,

$$\text{curl} = *d \quad \text{and} \quad \text{div} = *d*.$$

Note that this points to a way to unify the various versions of the fundamental theorem of calculus for surfaces and threefolds (namely, the theorems of Green, Stokes and Gauss). One can also use this notation to express Maxwell's equations quite efficiently. But these are both other stories.

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