MINIMAL SURFACES

Definition 0.1. We say that $S \subset \mathbb{R}^3$ is a *minimal surface* if it is a critical point for area.

We consider a particular class of minimal surfaces, minimal graphs, in what follows.

Let $u(x,y)$ be a graph of a surface $S \subset \mathbb{R}^3$ with $\Pi(S) = R$ and $u \in C^2(R)$. We know $Area(S) = \int\int_R \sqrt{1+|\nabla u|^2}dxdy$. Now we determine what it means for S to be a critical point for area. Consider any $v : R \to \mathbb{R}$ such that v is continuously differentiable and $v = 0$ on ∂R. Then the function $u_t = u + tv : R \to \mathbb{R}$ and $u_t(\partial R) = \partial S$ for all t. Denote $S_t = u_t(R)$. We say S is a critical point for area if

$$\frac{d}{dt}|_{t=0} Area(S_t) = 0.$$

Thus S is a critical point for area iff

$$\frac{d}{dt}|_{t=0} \int\int_R \sqrt{1+|\nabla u_t|^2}dxdy = 0.$$

But notice that $\nabla u_t = \nabla u + t\nabla v$ so $|\nabla u_t|^2 = |\nabla u|^2 + 2t\langle \nabla u, \nabla v\rangle + t^2|\nabla v|^2$. So

$$\frac{d}{dt}\sqrt{1+|\nabla u_t|^2} = \frac{\langle \nabla u, \nabla v\rangle + t|\nabla v|^2}{\sqrt{1+|\nabla u_t|^2}}.$$

Evaluating at $t = 0$ we get

$$\frac{\langle \nabla u, \nabla v\rangle}{\sqrt{1+|\nabla u|^2}}.$$

Now we can interchange the limit and the integral because v has continuous derivatives on R and thus as $t \to 0$, $\nabla u_t \to \nabla u$ uniformly on R. Thus S is a critical point for area if and only if for all $v \in C_0^1(R)$,

$$\int\int_R \frac{\langle \nabla u, \nabla v\rangle}{\sqrt{1+|\nabla u|^2}}dxdy = 0. \tag{1}$$

Now recall

$$\int_{\partial R} F \cdot n ds = \int\int_R (\partial F_1/\partial x + \partial F_2/\partial y)dydx = \int\int_R div(F)dxdy \tag{2}$$

where n is the normal to the boundary of ∂R. Set $F = v\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$; then $\int_{\partial R} F\cdot n ds = 0$ (since $v \equiv 0$ on the boundary). Now, we compute $div(F)$:

$$div\left(v\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = v_x\frac{u_x}{\sqrt{1+|\nabla u|^2}} + v_y\frac{u_y}{\sqrt{1+|\nabla u|^2}} + v\ div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

or

$$div(F) = \frac{\langle \nabla u, \nabla v\rangle}{\sqrt{1+|\nabla u|^2}} + div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

Using (1) and (2) we see for all $v \in C_0^1(R)$,

$$0 = \int\int_R v\ div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) dxdy.$$

Theorem 0.2. *Let $u(x,y)$ be a graph of a surface $S \subset \mathbb{R}^3$ with $\Pi(S) = R$ and $u \in C^2(S)$. Then S is a minimal surface if and only if*

$$div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

Proof. Most of our work is already done. We know that $\int\int_R v\ div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) dxdy = 0$ for all $v \in C^1$ with $v = 0$ on the boundary of R. Now suppose there exists $(x', y') \in R$ such that

$$div\left(\frac{\nabla u(x',y')}{\sqrt{1+|\nabla u(x',y')|^2}}\right) > 0.$$

Since $u \in C^2(R)$, it follows that there exists a neighborhood of (x', y'), $U \subset R$, such that $div\left(\frac{\nabla u(x,y)}{\sqrt{1+|\nabla u(x,y)|^2}}\right) > 0$ for all $(x,y) \in U$. Now choose $v \in C^1(R)$ such that $v = 0$ on $R\backslash U$ and $v > 0$ in U. But then

$$\int\int_R v\ div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) dxdy = \int\int_U v\ div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) dxdy > 0$$

which provides a contradiction. □