PSET 7 - DUE MARCH 31

1. 9.8:7 (6 points) Hint: It might help to define a scalar field $F(x, y, z) = f(u(x, y, z), v(x, y, z))$ where u, v are as needed.

2. Let $\mathbf{f} : \mathbb{R}^{m+n} \to \mathbb{R}^m$ be continuously differentiable and let $\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m$. Denote by $D\mathbf{f} = D\mathbf{f}^x + D\mathbf{f}^y$ the decomposition of the Jacobian such that for $\mathbf{h} \in \mathbb{R}^n, \mathbf{k} \in \mathbb{R}^m$, $D\mathbf{f}(\mathbf{x}, \mathbf{y})(\mathbf{h}, \mathbf{k}) = D\mathbf{f}^x(\mathbf{x}, \mathbf{y})\mathbf{h} + D\mathbf{f}^y(\mathbf{x}, \mathbf{y})\mathbf{k}$. (That is $D\mathbf{f}^x(\mathbf{x}, \mathbf{y}) : \mathbb{R}^n \to \mathbb{R}^m$ and $D\mathbf{f}^y(\mathbf{x}, \mathbf{y}) : \mathbb{R}^m \to \mathbb{R}^m$ as mentioned in class.)
Suppose for $\mathbf{a} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m$, $\mathbf{f}(\mathbf{a}, \mathbf{b}) = 0$ and $det(D\mathbf{f}^y(\mathbf{a}, \mathbf{b})) \neq 0$. We consider a few steps of the implicit function theorem in this setting. Let $F(\mathbf{x}, \mathbf{y}) : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ such that $F(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{f}(\mathbf{x}, \mathbf{y}))$.

- Write down the matrix DF as we described it in class. You may write it in block decomposition, but also explain how you produce each block! (2 points)
- Prove DF is invertible at $(\mathbf{a}, \mathbf{b})$. (3 points)
- Using the inverse function theorem, we know there exists $(\mathbf{a}, \mathbf{b}) \in V$ open and $F(\mathbf{a}, \mathbf{b}) \in W$ open such that $F : V \to W$ is invertible with continuously differentiable inverse G. Let $U = \{\mathbf{x} \in \mathbb{R}^n | (\mathbf{x}, \mathbf{0}) \in W\}$. Prove that U is open. (3 points)
- BONUS 1: Prove the existence of a well defined $\mathbf{g} : U \to \mathbb{R}^m$ such that $\mathbf{f}(\mathbf{x}, \mathbf{g}(\mathbf{x})) = \mathbf{0}$ for all $\mathbf{x} \in U$ and show this $\mathbf{g}$ is differentiable at $\mathbf{a}$. (6 points)
- BONUS 2: Prove the formula $D\mathbf{g}(\mathbf{a}) = -D\mathbf{f}^y(\mathbf{a}, \mathbf{b})^{-1} D\mathbf{f}^x(\mathbf{a}, \mathbf{b})$. (3 points)

3. 9.13:17 - part (a) should be a sketch on the (x, y)-plane (8 points)

4. 9.15:8,13 (8 points)

18.024 Multivariable Calculus with Theory
Spring 2011