Solutions for PSet 10

1. (E7:1,2)
   1 Let l be the arclength of C, and parameterize C by its arclength: $\alpha(t) = (x(t), y(t))$. Then $\alpha'(t) = 1$ thus $\mathbf{n}(t) = (y'(t), -x'(t))$.
   We have

$$\int_C \mathbf{f} \cdot \mathbf{n}\, ds = \int_0^l (P(\alpha(t)), Q(\alpha(t))) \cdot (y'(t), -x'(t))\ dt = \int_C -Q\ dx + P\ dy$$

   2 Using Green's Theorem for the function $\mathbf{g} = (-Q, P)$

$$\int\int_R \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right]\ dx\, dy = \int_C -Q\ dx + P\ dy$$

   But part(1) above gives us the value of the RHS. Combining, we have

$$\int\int_R \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right]\ dx\, dy = \int_C \mathbf{f} \cdot \mathbf{n}\, ds$$

2. (11.22:2) Q can be written as the sum of two functions $Q = Q_1 + Q_2$, where $Q_1 = -x^2 y e^{-y^2}$ and $Q_2 = \dfrac{1}{x^2 + y^2}$. Thus, the integral to evaluate is

$$\int_C P\ dx + Q_1\ dy + Q_2\ dy.$$

   Let R_1 be the square $\{|x| \leq a, |y| \leq a\}$ and C its boundary. First note that

$$\frac{\partial P}{\partial y} = -2yxe^{-y^2} = \frac{\partial Q_1}{\partial x}$$

   Then, by Green's theorem:

$$\int_C P\, dx + Q_1\, dy = \int\int_{R_1} \left[\frac{\partial P}{\partial y} - \frac{\partial Q_1}{\partial x}\right] dx\, dy = 0.$$

   To compute the remaining part $\int_C Q_2\, dy$, first observe that the integral will certainly be zero along $y = \pm a$ as there one has $dy \equiv 0$. So, we can compute

$$\int_C Q_2\, dy = \int_{-a}^{a} \frac{dt}{a^2 + t^2} - \int_{-a}^{a} \frac{dt}{a^2 + t^2} = 0.$$

Notice the sign on the second integral corresponds to the fact that the direction of the parameterized curve is opposite the counterclockwise orientation on the square.

3. (11.22:4) Note that

$$\mathbf{f}\cdot\mathbf{g} = v\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) + u\left(\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right) = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(uv) - \frac{\partial}{\partial y}(uv)$$

Applying Green's Theorem for the function $\mathbf{h} = (uv, uv)$:

$$\int\int_R \mathbf{f}\cdot\mathbf{g}\,dx\,dy = \int\int_R \left[\frac{\partial}{\partial x}(uv) - \frac{\partial}{\partial y}(uv)\right]\,dx\,dy = \int_C uv\,(dx+dy) = \int_C (1)(y)\,(dx+dy)$$

Parameterize the circle by $\mathbf{s}(t) = (\cos t, \sin t)$ then

$$\int\int_R \mathbf{f}\cdot\mathbf{g}\,dx\,dy = \int_C y\,(dx+dy) = \int_0^{2\pi} \sin t(-\sin t + \cos t)\,dt = -\pi$$

4. (11.22:8) If C is parameterized by arclength $s(t) = (x(t), y(t))$ then $\mathbf{n} = (y', -x')$

   (a) We can write

$$\begin{aligned}\int_C \frac{\partial g}{\partial n}\,ds &= \int_C \nabla g\cdot\mathbf{n}\,ds = \\ \int_C \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)\cdot\left(\frac{\partial y}{\partial t}, \frac{\partial x}{\partial t}\right)\,dt &= \int_C \left[\frac{\partial g}{\partial x}\frac{\partial y}{\partial t} - \frac{\partial g}{\partial y}\frac{\partial x}{\partial t}\right]\,dt = \int_C \frac{\partial g}{\partial x}\,dy - \frac{\partial g}{\partial y}\,dx\end{aligned}$$

   Using Green's Theorem we get

$$\int\int_R \left[\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right]\,dx\,dy = \int\int_R \nabla^2 g\,dx\,dy$$

   (b) Similarly

$$\begin{aligned}\int_C f\frac{\partial g}{\partial n}\,ds &= \int_C f\frac{\partial g}{\partial x}\,dy - f\frac{\partial g}{\partial y}\,dx \\ &= \int\int_R \left[\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial y^2}\right]\,dx\,dy \\ &= \int\int_R \left[\nabla f\cdot\nabla g + f\nabla^2 g\right]\,dx\,dy\end{aligned}$$

(c) To prove this part, apply part (b) with the roles of f and g reversed. Then subtract this equation from the equation stated in part (b).

5. (11.25:3) Whenever C_1 and C_2 cobound a region $R_1 \subset R$ we have

$$\int_{C_1} P\,dx + Q\,dy - \int_{C_2} P\,dx + Q\,dy = \int\int_{R_1} \left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] dx\,dy$$

This evaluates to 0 when $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. Thus

$$\int_{C_1} P\,dx + Q\,dy = \int_{C_2} P\,dx + Q\,dy$$

Let C be a Jordan curve, oriented counterclockwise that does not contract in the annulus. Since one cannot apply Green's Theorem on this region $\int_C P\,dx + Q\,dy = A$ for some $A \in \mathbb{R}$, not necessarily zero. For example when $(P,Q) = (\dfrac{x}{x^2+y^2}, -\dfrac{y}{x^2+y^2})$ on $R = \{1 \le x^2+y^2 \le 2\}$, we have $\int_C P\,dx + Q\,dy = 2\pi$ (as we will see in (11.25:1)). Now any Jordan curve C' in R either bounds a simply connected region or it cobounds a region with C or $-C$. (We are somewhat glossing over the difficulty that C, C' might intersect. But this problem can be easily rectified by choosing a third curve that cobounds with both C and C' and intersects neither of them.) In the first case, using Green's Theorem implies

$$\int_{C'} P\,dx + Q\,dy = \int\int_D \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0$$

where here $D \subset R$ and $\partial D = C'$.

In the second case, we use the observations outlined initially to determine

$$\int_{C'} P\,dx + Q\,dy = \pm \int_C P\,dx + Q\,dy = \pm A.$$

Thus there are 3 possible values for line integrals along piecewise smooth Jordan curves.

6. (11.25:1)

(a) First let B_r (boundary of disc D_r) be a circle around $(0,0)$ with radius r. We can parameterize B_r by $\mathbf{s}(t) = (r\cos t, r\sin t)$. Then $P = r\sin t/r^2$ and $Q = -r\cos t/r^2$. So

$$\begin{aligned}\int_{B_r} P\,dx + Q\,dy &= \int_0^{2\pi} (r\sin t/r^2)(-r\sin t)\,dt + (-r\cos t/r^2)(r\cos t)\,dt \\ &= \int_0^{2\pi} \frac{-r^2}{r^2}\,dt = -2\pi\end{aligned}$$

For any piecewise smooth Jordan curve C that bounds a region R that contains $(0,0)$ we can choose r small enough so that the disc D_r of radius r lies inside C. Then $R - D_r$ is a region where $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and these partial derivatives are well defined. Thus, as in the previous exercise we have that

$$\int_C P\,dx + Q\,dy = \pm \int_{B_r} P\,dx + Q\,dy = \pm 2\pi.$$

The sign depends on the orientation of C with respect to the orientation of B_r. The integral is positive if C is oriented clockwise and negative otherwise.

(b) If $(0,0)$ is outside the region then in the whole region $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (and these partial derivatives are everywhere well defined). Thus we can once more use Green's Theorem to see that $\int_C P\,dx + Q\,dy = 0$.