Solutions for PSet 11

1. (11.28:14) We substitute $u = x - y$ and $v = x + y$. the Jacobian of the transformation $(x, y) \mapsto (u, v)$ is

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

This has determinant 2 and the vertices of the parallelogram S correspond to $-\pi < u < \pi, \pi < v < 3\pi$. Thus:

$$\int\int_S (x-y)^2 \sin^2(x+y)\, dx\, dy = \int_{\pi}^{3\pi} \int_{-\pi}^{\pi} \frac{1}{2} u^2 \sin^2 v\, du\, dv$$

This simplifies to:

$$\frac{1}{2}\left[\frac{v}{2} - \frac{\sin(2v)}{4}\right]_{v=\pi}^{3\pi} \left[\frac{u^3}{3}\right]_{u=-\pi}^{\pi} = \frac{\pi^4}{3}$$

2. (11.28:16)

(a) We can apply Fubini's Theorem:

$$\int\int_R e^{-(x^2+y^2)}\, dx\, dy = \int_{-r}^{r} e^{-x^2}\, dx \int_{-r}^{r} e^{-y^2}\, dy = (I(r))^2$$

(b) $C_1 \subset R \subset C_2$ and the function $e^{-(x^2+y^2)} > 0$ thus

$$\int\int_R e^{-(x^2+y^2)}\, dx\, dy - \int\int_{C_1} e^{-(x^2+y^2)}\, dx\, dy = \int\int_{R \setminus C_1} e^{-(x^2+y^2)}\, dx\, dy > 0$$

and

$$\int\int_{C_2} e^{-(x^2+y^2)}\, dx\, dy - \int\int_{R} e^{-(x^2+y^2)}\, dx\, dy = \int\int_{C_2 \setminus R} e^{-(x^2+y^2)}\, dx\, dy > 0$$

Combining the two results:

$$\int\int_{C_1} e^{-(x^2+y^2)}\, dx\, dy < \int\int_R e^{-(x^2+y^2)}\, dx\, dy < \int\int_{C_2} e^{-(x^2+y^2)}\, dx\, dy$$

thus proving the required statement.

(c) For a disc C of radius s

$$\int\int_C e^{-(x^2+y^2)}\,dx\,dy \;=\; \int_0^{2\pi}\int_0^s e^{-\rho^2}\rho\,d\rho\,d\vartheta =$$
$$2\pi\int_0^{s^2}\frac{1}{2}e^{-u}\,du \;=\; \pi\left[-e^{-u}\right]_{u=0}^{u=s^2} = \pi(1-e^{-s^2})$$

For the circles C_1 (inscribing square of side $2r$) and C_2 (circumscribing square of side $2r$), the radii are r and $\sqrt{2}r$ respectively. Substituting to (b) we get:

$$\pi(1-e^{-r^2}) < I^2(r) < \pi(1-e^{-2r^2})$$

as $r \to \infty$:

$$\pi \leq \lim_{r\to\infty} I^2(r) \leq \pi$$

Thus $\lim_{r\to\infty} I(r)$ exists and equals $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. In other words, $\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

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