Solution for PSet 2

1. (a) First, note $T_\theta(1,0) = (\cos\theta, \sin\theta)$ and $T_\theta(0,1) = (\cos(\theta+\pi/2), \sin(\theta+\pi/2)) = (-\sin\theta, \cos\theta)$ so the matrix is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   (b) There are two ways to determine this problem, but perhaps the easiest is to find $T_{-\theta}$. In that case $T_{-\theta}(1,0) = (\cos(-\theta), \sin(-\theta)) = (\cos\theta, -\sin\theta)$ and $T_{-\theta}(0,1) = (\cos(\pi/2-\theta), \sin(\pi/2-\theta)) = (\sin\theta, \cos\theta)$. So

$$T_\theta^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

   Finally, to check we note

$$TT^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix}$$

   This evaluates to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. First note that $T(1,0,0) = (1,0) = 1\cdot(1,0) + 0\cdot(1,-1)$; $T(1,1,0) = (1,1) = 2\cdot(1,0) - 1\cdot(1,-1)$; $T(1,1,1) = (1,1) = 2\cdot(1,0) - 1\cdot(1,-1)$ and thus the matrix for T in these bases is

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix}.$$

   To find the matrix for S we perform the same process:
   $S(1,0,0) = (-1,0,0) = -1\cdot(1,0,0) + 0\cdot(1,1,0) + 0\cdot(1,1,1)$.
   $S(1,1,0) = (-1,-1,0) = 0\cdot(1,0,0) - 1\cdot(1,1,0) + 0\cdot(1,1,1)$
   $S(1,1,1) = (-1,-1,-1) = 0\cdot(1,0,0) + 0\cdot(1,1,0) - 1\cdot(1,1,1)$
   Thus the matrix for this transformation is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus we find the matrix for TS by multiplication:

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix}.$$

3. (2.20:9) The row reduced forms are shown below (without elaborating each step involved). First, the augmented matrix and the first few reductions:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & a & 6 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 6 & 10-a & 4 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1/3 & 2/3 \\ 0 & & 8-a & 0 \end{array}\right)$$

Now if $8-a \neq 0$ then we can divide the last row by $8-a$ and simplify:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2/3 \\ 0 & & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2/3 \\ 0 & & 1 & 0 \end{array}\right)$$

to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

and thus the unique solution is $x = 4/3, y = 2/3, z = 0$.

Now if $a = 8$, then z is a free variable and we reduce

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1/3 & 2/3 \\ 0 & & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 5/3 & 4/3 \\ 0 & 1 & 1/3 & 2/3 \\ 0 & & 0 & 0 \end{array}\right)$$

and thus $x + 5/3z = 4/3$ and $y + 1/3z = 2/3$. So solutions are of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4/3 \\ 2/3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 5/3 \\ 1/3 \\ -1 \end{pmatrix}.$$

4. (a) First, if λ is an eigenvalue for A then there exists $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. That is $A\mathbf{x} - \lambda I_n\mathbf{x} = \mathbf{0}$ or $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ for $\mathbf{x} \neq \mathbf{0}$. Thus, the null space of $A - \lambda I_n$ has positive dimension and thus $A - \lambda I_n$ is not an invertible matrix. This implies $det(A - \lambda I_n) = 0$.

Now, going in the reverse direction, if $det(A - \lambda I_n) = 0$ then null space $N(A - \lambda I_n)$ contains a non-zero vector. That is, there exists $\mathbf{x}$ such that $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$. But this exactly corresponds to $A\mathbf{x} = \lambda I_n\mathbf{x} = \lambda\mathbf{x}$.

(b) Consider the matrix

$$A - \lambda I_3 = \begin{pmatrix} 4-\lambda & 1 & -2 \\ 16 & -2-\lambda & -8 \\ 4 & -2 & -2-\lambda \end{pmatrix}.$$

A tedious calculation gives $det(A - \lambda I_3) = 36\lambda - \lambda^3 = \lambda(36 - \lambda^2) = \lambda(6 - \lambda)(6 + \lambda)$. This is zero precisely when $\lambda = 0, -6, 6$ and thus these are the eigenvalues for the matrix A.

(c) Since 0 is an eigenvalue, there exists $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = 0 \cdot \mathbf{x} = \mathbf{0}$. Thus, the null space of A is non-trivial. This immediately implies A is not invertible.

5. $X^3 = Y^3$ and $X^2Y = Y^2X$ taken together allow us to write

$$(X^2 + Y^2)X = X^3 + Y^2X = Y^3 + X^2Y = (X^2 + Y^2)Y.$$

Notice that if $X \neq Y$ then $X^2 + Y^2$ cannot be invertible. Thus, a necessary condition is that $X = Y$.