Solutions for PSet 4

1. (B63:3)

$$f(t) = \begin{cases} (t, t\cos\left(\frac{\pi}{t}\right)) & \text{for } 0 < t \leq 1, \\ (0,0) & \text{for } t = 0. \end{cases}$$

$f : \mathbb{R} \to \mathbb{R}^2$, thus it is continuous if both $f_1(t) = t$ and

$$f_2(t) = \begin{cases} t\cos\left(\frac{\pi}{t}\right) & \text{for } 0 < t \leq 1, \\ 0 & \text{for } t = 0. \end{cases}$$

are continuous. f_1 is clearly continuous at any point t, and so is f_2 at any point $t \neq 0$. For $t = 0$ we have to check that if $t_n \to 0$, then $f_2(t_n) \to f_2(0) = 0$:

$$|f_2(t_n)| = |t_n \cos\left(\frac{\pi}{t_n}\right)| \leq |t_n|$$

thus it tends to 0 too. Further, note that both $f_1(t)$ and $f_2(t)$ do not have any self intersections and are therefore simple curves.

Because $f(t)$ is simple and continuous over a continuous stretch of t, we can assess whether $f(t)$ has a finite arc length (is rectifiable) by:

(a) partitioning the space of t into n discrete blocks defined by vertices t_1, t_2, $\ldots t_n$.

(b) defining a polygonal arc connecting points $(t_1, f(t_1))$, $(t_2, f(t_2))$, $\ldots (t_n, f(t_n))$ - this represents a sampled approximation of $f(t)$

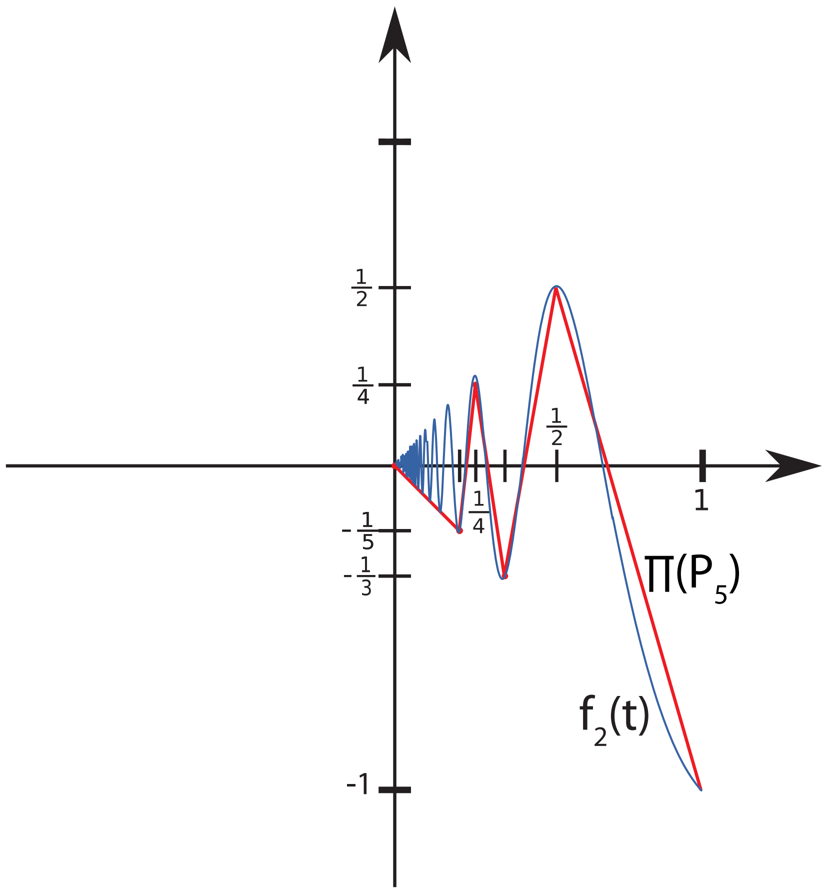
(c) considering the limit as $n \to \infty$ of the length of this polygonal arc

For $n = 5$, the partition of t is defined by the collection of points t' in P_5:

$$P_5 = \left\{0, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$$

and the polygonal arc connects the points $(t', t'\cos\left(\frac{\pi}{t'}\right))$:

$$\left\{(0,0), \left(\frac{1}{5}, -\frac{1}{5}\right), \left(\frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{3}, -\frac{1}{3}\right), \left(\frac{1}{2}, \frac{1}{2}\right), (1,-1)\right\}.$$



Notice

$$|\pi(P_n)| = ||(1/n, (-1)^n/n) - (0,0)|| + \sum_{k=2}^{n} \sqrt{\left(\frac{1}{k} - \frac{1}{k-1}\right)^2 + \left(\frac{(-1)^k}{k} - \frac{(-1)^{k-1}}{k-1}\right)^2}$$

$$= \frac{\sqrt{2}}{n} + \sum_{k=2}^{n} \sqrt{\left(\frac{1}{k(k-1)}\right)^2 + \left(\frac{(-1)^k 2k + (-1)^{k+1}}{k(k-1)}\right)^2}$$

$$= \frac{\sqrt{2}}{n} + \sum_{k=2}^{n} \frac{1}{k(k-1)} \sqrt{4k^2 - 4k + 2} > \frac{\sqrt{2}}{n} + \sum_{k=2}^{n} \frac{2k-1}{k(k-1)} = \frac{\sqrt{2}}{n} + \sum_{k=2}^{n} \frac{1}{(k-1)} + \frac{1}{k} > \frac{\sqrt{2}}{n} + \sum_{k=2}^{n} \frac{2}{k}.$$

(This is not quite what the notes have, but nearly so.)

Thus the length of P_n tends to ∞ as n goes to ∞, so $f(t)$ cannot be rectifiable.

2. (14.13:21) By the chain-rule the derivative of $Y(t) = X[u(t)]$ is $Y'(t) = u'(t)X'[u(t)]$. Using this:

$$\int_c^d |Y'(t)|dt = \int_c^d |u'(t)X'[u(t)]|dt$$

Substituting $u = u(t)$

$$\int_c^d |Y'(t)|dt = \int_{u(c)}^{u(d)} \frac{|u'(t)X'[u(t)]|}{u'(t)} du$$

But $u'(t) > 0$, as a re-parametrization of a curve is by definition an increasing function (and further by assumption it is strictly increasing). Therefore $\frac{|u'(t)|}{u'(t)} = 1$ and

$$\int_c^d |Y'(t)|dt = \int_{u(c)}^{u(d)} |X'(u)|\, du$$

3. (14.15:11)

(a) With the notation of the exercise

$$\mathbf{v}(t) = 5(\cos\alpha(t)\mathbf{i} + \sin\alpha(t)\mathbf{j})$$

and

$$\mathbf{a}(t) = \mathbf{v}'(t) = 5\alpha'(t)(-\sin\alpha(t)\mathbf{i} + \cos\alpha(t)\mathbf{j})$$

then

$$\kappa(t) \equiv \frac{|\mathbf{a}(t) \times \mathbf{v}(t)|}{v(t)^3} = \frac{|25\alpha'(t)|}{125} = 2t$$

This means:

$$|\alpha'(t)| = 10t.$$

As a result, $\alpha'(t)$ could be $-10t$ or $10t$. Since $\alpha(0) = \frac{\pi}{2}$ and the curve stays in the positive half plane, i.e. $\alpha(t) < \frac{\pi}{2}\, \forall t$, we see that $\alpha'(0) < 0$.

Then by continuity $\alpha'(t) < 0$. Thus the correct solution is $\alpha'(t) = -10t$. Integrating we get $\alpha(t) = -5t^2 + C$. As $\alpha(0) = C$, $C = \frac{\pi}{2}$ and we get

$$\alpha(t) = -5t^2 + \frac{\pi}{2}.$$

(b) We have already computed everything necessary:

$$\mathbf{v}(t) = 5(\cos\alpha(t)\mathbf{i} + \sin\alpha(t)\mathbf{j}) = 5(\cos(-5t^2 + \frac{\pi}{2})\mathbf{i} + \sin(-5t^2 + \frac{\pi}{2})\mathbf{j})$$

4. Assume $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous. Given an open set $U \subset \mathbb{R}^m$ we want to prove that $f^{-1}(U)$ is open. This means that for any point $x \in f^{-1}(U)$ we need to show there exists $R > 0$ such that $B_R(x) \subset f^{-1}(U)$.

   Given $x \in f^{-1}(U)$, we know $f(x) \in U$. As U is open, there exists $r > 0$ such that $B_r(f(x)) \subset U$. Recall f is continuous at x, thus for any $\epsilon > 0$ there is a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Choosing $\epsilon = r$, find δ. Then continuity tells us

$$f(B_\delta(x)) \subset B_r(f(x)) \subset U.$$

   It follows immediately that $B_\delta(x) \subset f^{-1}(U)$. Thus, $f^{-1}(U)$ is open.

5. (8.5:2,4) **8.5:2** Let

$$L := \lim_{(x,y)\to(a,b)} f(x,y).$$

   We would like to prove that if

$$\lim_{x\to a}(f(x,y)) \text{ exists for every } y,$$

   then

$$\lim_{y\to b}\lim_{x\to a}(f(x,y)) = L$$

   Let us denote $\lim_{x\to a} f(x,y) = L_y$ for each y. Then we need to show $\lim_{y\to b} L_y = L$. Suppose not. Then for some $\epsilon > 0$ there exists a sequence $y_n \to b$ such that $|L_{y_n} - L| > \epsilon$. Reindex the sequence y_n, and perhaps remove some elements, so that $|y_n - b| < 1/n$ for each n. Now, for each n choose x_n such that $|x_n - a| < 1/n$ and

$$|f(x_n, y_n) - L_{y_n}| < \epsilon/2.$$

Now consider for each n,

$$|f(x_n, y_n) - L| \geq |L - L_{y_n}| - |L_{y_n} - f(x_n, y_n)| > \epsilon - \epsilon/2 = \epsilon/2 > 0.$$

Thus, we constructed a sequence $(x_n, y_n) \to (a, b)$ such that $f(x_n, y_n)$ does not converge to L. This implies a contradiction and it follows that one must have $\lim_{y\to b} L_y \to L$.

8.5:4

$$f(x,y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2} \text{ whenever } x^2y^2 + (x-y)^2 \neq 0$$

Then

$$\lim_{y\to 0}\lim_{x\to 0}(f(x,y)) = \lim_{y\to 0} 0 = 0,$$

and similarly

$$\lim_{x\to 0}\lim_{y\to 0}(f(x,y)) = \lim_{x\to 0} 0 = 0.$$

But for $y = x$

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^4}{x^4} = 1.$$

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18.024 Multivariable Calculus with Theory
Spring 2011

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