Solutions for PSet 7

1. (9.8:7) Hint: It might help to define a scalar field $F(x,y,z) = f(u(x,y,z), v(x,y,z))$ where u, v are as needed. We first assume that $x \neq 0$. Given $g(x,y) = z$, we know

$$\frac{\partial g}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}; \qquad \frac{\partial g}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}.$$

Now, we need only use the chain rule to determine the result. First observe that $\nabla u = (-y/x^2, 1/x, 0), \nabla v = (-z/x^2, 0, 1/x)$. Now we compute

$$\frac{\partial F}{\partial x} = \nabla f(u,v) \cdot (\partial u/\partial x, \partial v/\partial x) = D_1 f(u,v)(-y/x^2) + D_2 f(u,v)(-z/x^2),$$

$$\frac{\partial F}{\partial y} = \nabla f(u,v) \cdot (\partial u/\partial y, \partial v/\partial y) = D_1 f(u,v)(1/x),$$

$$\frac{\partial F}{\partial z} = \nabla f(u,v) \cdot (\partial u/\partial z, \partial v/\partial z) = D_2 f(u,v)(1/x).$$

An easy computation then gives

$$\frac{\partial g}{\partial x} = \frac{y D_1 f(u,v)}{x D_2 f(u,v)} + \frac{z}{x}; \qquad \frac{\partial g}{\partial y} = \frac{-D_1 f(u,v)}{D_2 f(u,v)}.$$

Thus

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = z = g(x,y).$$

2. (a) First, DF has the form of a block matrix.

$$DF(\mathbf{x},\mathbf{y}) = \left(\begin{array}{c|c} I_n & 0 \\ \hline D\mathbf{f}^x(\mathbf{x},\mathbf{y}) & D\mathbf{f}^y(\mathbf{x},\mathbf{y}) \end{array}\right) \tag{1}$$

This comes when we consider

$$F(\mathbf{x},\mathbf{y}) = (F_1(\mathbf{x},\mathbf{y}), \ldots, F_n(\mathbf{x},\mathbf{y}), f_1(\mathbf{x},\mathbf{y}), \ldots, f_m(\mathbf{x},\mathbf{y}))$$

where here $F_i(\mathbf{x},\mathbf{y}) = \mathbf{x} \cdot \mathbf{e}_i$. Then $\frac{\partial F_i}{\partial x_j} = \delta_{ij}$ for $1 \leq i, j \leq n$ and $\frac{\partial F_i}{\partial y_k} = 0$ for $1 \leq i \leq n, 1 \leq k \leq m$. The bottom portion of the matrix is exactly what we get based on our determination of $D\mathbf{f}^x, D\mathbf{f}^y$.

(b)
$$DF(\mathbf{a},\mathbf{b}) = \left(\begin{array}{c|c} I_n & 0 \\ \hline D\mathbf{f}^x(\mathbf{a},\mathbf{b}) & D\mathbf{f}^y(\mathbf{a},\mathbf{b}) \end{array}\right) \tag{2}$$

The invertibility of Df^y gives that DF is invertible at $(\mathbf{a},\mathbf{b})$. That is, recall that an invertible matrix has a row reduction that reduces it to the identity matrix. Using this particular row reduction, reduce the bottom m rows of DF. The new matrix $\tilde{DF}(\mathbf{a},\mathbf{b})$ is lower triangular (everything above the main diagonal is zero). Recall in this case that $det(\tilde{DF}(\mathbf{a},\mathbf{b})) = 1$ and since row reduction operations preserves the non-zero property of the determinant, $det(DF(\mathbf{a},\mathbf{b})) = det(\tilde{DF}(\mathbf{a},\mathbf{b})) \neq 0$.

(c) Let $m : \mathbb{R}^n \to \mathbb{R}^{n+m}$ such that $m(\mathbf{x}) = (\mathbf{x},\mathbf{0})$. It's obvious that m is a continuous function and $m^{-1}(W) = U$. This, along with the fact that W is open (by definition), implies U is open.

(d) Now if $\mathbf{x} \in U$ then there exists $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{f}(\mathbf{x},\mathbf{y}) = 0$. Suppose there was a second $\mathbf{y}'$ such that $\mathbf{f}(\mathbf{x},\mathbf{y}') = 0$. But then $F(\mathbf{x},\mathbf{y}) = F(\mathbf{x},\mathbf{y}')$ and since F is one-to-one we know that $\mathbf{y} = \mathbf{y}'$.

We define $\mathbf{g} : U \to \mathbb{R}^m$ by this uniqueness, and by definition $\mathbf{f}(\mathbf{x},\mathbf{g}(\mathbf{x})) = 0$. Now for $\mathbf{x} \in U$, $F(\mathbf{x},\mathbf{g}(\mathbf{x})) = (\mathbf{x},\mathbf{0})$. Let G again be the inverse of F. Then $G(\mathbf{x},\mathbf{0}) = (\mathbf{x},\mathbf{g}(\mathbf{x}))$. Now notice for any $1 \leq k \leq n$,

$$G(\mathbf{x} + h\mathbf{e}_k, \mathbf{0}) - G(\mathbf{x},\mathbf{0}) = (\mathbf{x} + h\mathbf{e}_k, \mathbf{g}(\mathbf{x} + h\mathbf{e}_k)).$$

Thus, the differentiability of G at $(\mathbf{x},\mathbf{0})$ in the direction $\mathbf{e}_k$ for each $1 \leq k \leq n$ implies the differentiability of $\mathbf{g}$.

(e) Now we calculate the formula for the derivative:
Let $\Phi : \mathbb{R}^n \to \mathbb{R}^{n+m}$ such that $\Phi(\mathbf{x}) = (\mathbf{x},\mathbf{g}(\mathbf{x}))$. Then $D\Phi(\mathbf{x})\mathbf{h} = (\mathbf{h}, D\mathbf{g}(\mathbf{x})\mathbf{h})$. Now, for all $\mathbf{x} \in U$, $\mathbf{f}(\Phi(\mathbf{x})) = \mathbf{f}(\mathbf{x},\mathbf{g}(\mathbf{x})) = 0$ and thus

$$D\mathbf{f}(\mathbf{x},\mathbf{g}(\mathbf{x}))D\Phi(\mathbf{x}) \equiv 0.$$

Evaluating this at $\mathbf{x} = \mathbf{a}$ we get

$$D\mathbf{f}(\mathbf{a},\mathbf{b})D\Phi(\mathbf{a}) = 0.$$

Now note $D\mathbf{f}(\mathbf{x},\mathbf{y})D\Phi(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$ and so for a fixed $\mathbf{h} \in \mathbb{R}^n$ we get

$$0 = D\mathbf{f}(\mathbf{a},\mathbf{b})D\Phi(\mathbf{a})\mathbf{h} = D\mathbf{f}(\mathbf{a},\mathbf{b})(\mathbf{h}, D\mathbf{g}(\mathbf{x})\mathbf{h}) = D\mathbf{f}^x(\mathbf{a},\mathbf{b})\mathbf{h} + D\mathbf{f}^y(\mathbf{a},\mathbf{b})D\mathbf{g}(\mathbf{x})\mathbf{h}.$$

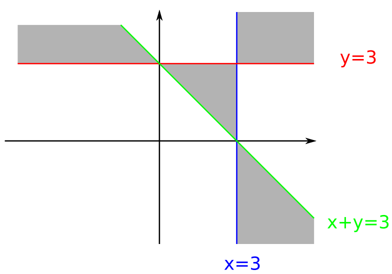
This gives the result.

3. (9.13:17)

(a) $f(x,y) = (3-x)(3-y)(x+y-3)$ vanishes at $x = 3$, $y = 3$, and $x+y = 3$. We have $f(x,y) > 0$ for each of the 4 conditions:

Box	$3 > x$	$3 > y$	$x + y > 3$
Box 1	yes	yes	yes
Box 2	yes	no	no
Box 3	no	yes	no
Box 4	no	no	yes

In other words, if both x and y are big, then $f(x,y) > 0$. The colored lines on the graph below indicate where the function $f(x,y)$ vanishes. Also $f(x,y)$ changes sign whenever we cross one of the colored lines, thus the gray areas indicate where $f(x,y) > 0$.



(b) The partial derivatives:

$$\begin{aligned} D_1 f(x,y) &= -1(3-y)(x+y-3)+(3-x)(3-y)1 = (y-3)(2x+y-6) \\ D_2 f(x,y) &= (3-x)(-1)(x+y-3)+(3-x)(3-y)1 = (x-3)(x+2y-6) \end{aligned}$$

Thus $D_1 f(x,y) = D_2(x,y) = 0$ at stationary points $(3,3),(3,0),(0,3)$ and $(2,2)$

(c) The second derivative matrix is:

$$Hf(x,y) = \begin{pmatrix} 2y-6 & 2x+2y-9 \\ 2x+2y-9 & 2x-6 \end{pmatrix}$$

Substituting at the stationary points:

$$Hf(3,3) = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}, \qquad Hf(3,0) = \begin{pmatrix} -6 & -3 \\ -3 & 0 \end{pmatrix},$$

$$Hf(0,3) = \begin{pmatrix} 0 & -3 \\ -3 & -6 \end{pmatrix} \quad \text{and} \quad Hf(2,2) = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$$

Thus:

(x,y)	$\mathrm{tr}(Hf(x,y))$	$\det(Hf(x,y))$	type of stationary point
$(3,3)$	0	-9	saddle
$(3,0)$	-6	-9	saddle
$(0,3)$	-6	-9	saddle
$(2,2)$	-4	3	relative or local maximum

The function has no relative minima.

(d) Setting $x = y$ the function $f(x,x)$ is a polynomial of degree 3, thus it can be arbitrarily large and arbitrarily small too, thus it has no maxima and no minima, nor does $f(x,y)$ in general.

4. (9.15:8,13) **9.15:8** Let $g_1, g_2 : \mathbb{R}^3 \to \mathbb{R}$ be defined as

$$\begin{aligned} g_1(x,y,z) &= x^2 - xy + y^2 - z^2 - 1; \\ g_2(x,y,z) &= x^2 + y^2 - 1 \end{aligned}$$

The surfaces in question are $g_1(x,y,z) = 0$ and $g_2(x,y,z) = 0$. We would like to minimize the distance to the origin, defined by the function $f(x,y,z) = x^2 + y^2 + z^2$ on the surfaces $\{g_1(x,y,z) = g_2(x,y,z) = 0\}$.

By the method of Lagrange multipliers there must be constants λ_1 and λ_2 such that:

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

That is:

$$(2x, 2y, 2z) = \lambda_1(2x - y, 2y - x, -2z) + \lambda_2(2x, 2y, 0)$$

This leaves us with 5 equations:

$$\begin{aligned}
(2\lambda_1 + 2\lambda_2 - 2)x - \lambda_1 y &= 0 \\
(2\lambda_1 + 2\lambda_2 - 2)y - \lambda_1 x &= 0 \\
-(2\lambda_1 + 2)z &= 0 \\
x^2 - xy + y^2 - z^2 - 1 &= 0 \\
x^2 + y^2 - 1 &= 0
\end{aligned}$$

By a straightforward case analysis the solutions are $(x, y, z) = (1, 0, 0), (0, 1, 0), (-1, 0, 0), (0, -1, 0)$. The distance at each of these points is 1.

9.15:13 In this problem

$$g(x, y) = x^2 + 4y^2 - 4$$

The distance from the point (x, y) to the line $x + y = 4$ is

$$f(x, y) = \frac{|x + y - 4|}{\sqrt{2}}$$

Note that for any point (x, y) on the ellipse, $x + y - 4 < 0$, thus $|x + y - 4| = 4 - x - y$.

Using Lagrange multipliers at the extrema points there is a λ such that:

$$\nabla f = \lambda \nabla g$$

That is

$$(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \lambda(2x, 8y)$$

Thus we have three equations:

$$\begin{aligned}
-\frac{1}{\sqrt{2}} &= 2\lambda x \\
-\frac{1}{\sqrt{2}} &= 8\lambda y \\
x^2 + 4y^2 - 4 &= 0
\end{aligned}$$

The solutions are $(x, y) = \pm(\frac{4}{\sqrt{5}}, \frac{1}{\sqrt{5}})$. Evaluating $f(x, y)$ at these solutions, we find the greatest distance is $\frac{4+\sqrt{5}}{\sqrt{2}}$ and the least distance is $\frac{4-\sqrt{5}}{\sqrt{2}}$.

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