Solutions for PSet 8

1. (10.5:11) Parameterize the sides of the square C by maps $s_i : [0,1] \to \mathbb{R}^2$ by

$$\begin{aligned} s_1(t) &= (1-t, t); \\ s_2(t) &= (-t, 1-t); \\ s_3(t) &= (t-1, -t); \\ s_4(t) &= (t, t-1). \end{aligned}$$

   With this parametrization:

$$\int_C \frac{\mathrm{d}x + \mathrm{d}y}{|x|+|y|} = \int_0^1 \frac{-1+1}{(1-t)+t}\mathrm{d}t + \int_0^1 \frac{-1-1}{t+(1-t)}\mathrm{d}t + \int_0^1 \frac{1-1}{(1-t)+t}\mathrm{d}t + \int_0^1 \frac{1+1}{t+(1-t)}\mathrm{d}t$$

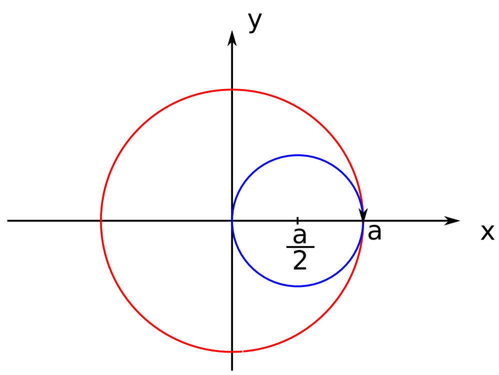
   The first and the third summands are 0, and the second and fourth terms cancel each other, giving:

$$\int_C \frac{\mathrm{d}x + \mathrm{d}y}{|x|+|y|} = 0$$

2. (10.9:6) Writing the equation of the cylinder in complete square form:

$$(x - \frac{a}{2})^2 + y^2 = \frac{a^2}{4}$$

   Thus looking from high above the xy-plane the picture looks like:



   The parametrization of the cylinders' intersection with the xy-plane is:

$$\widetilde{s}(t) = \left(\frac{a}{2}\cos t + \frac{a}{2}, \frac{a}{2}\sin t, 0\right)$$

We need to lift it up to sit on the sphere:

$$s(t) = \left(\frac{a}{2}\cos t + \frac{a}{2}, \frac{a}{2}\sin t, z(t)\right),$$

where $z(t) \geq 0$ and

$$\left(\frac{a}{2}\cos t + \frac{a}{2}\right)^2 + \left(\frac{a}{2}\sin t\right)^2 + z(t)^2 = a\left(\frac{a}{2}\cos t + \frac{a}{2}\right) + z(t)^2 = a^2$$

This means, that

$$z(t) = \frac{a}{\sqrt{2}}\sqrt{1-\cos t}$$

Now

$$\begin{aligned}
&\int_C (y^2, z^2, x^2) \cdot \mathrm{d}(x, y, z) \\
&= \int_0^{2\pi} \frac{a^3}{8}\left(\sin^2 t, 2(1-\cos t), (\cos t + 1)^2)\right) \cdot \left(-\sin t, \cos t, \frac{\sin t}{\sqrt{2(1-\cos t)}}\right) \mathrm{d}t \\
&= \frac{a^3}{8}\int_0^{2\pi} \left(-\sin^3 t + 2\cos t(1-\cos t) + \frac{\sin t(\cos t + 1)^2}{\sqrt{2(1-\cos t)}}\right) \mathrm{d}t \\
&= -\frac{a^3}{8}\int_0^{2\pi} \sin^3 t \mathrm{d}t + + \frac{a^3}{4}\int_0^{2\pi} \cos t(1-\cos t)\mathrm{d}t + \frac{a^3}{8}\int_0^{2\pi} \frac{\sin t(\cos t+1)^2}{\sqrt{2(1-\cos t)}}\mathrm{d}t
\end{aligned}$$

Computing each of the integrals separately we get:

$$= 0 + \frac{a^3}{4}\pi + 0 = \frac{a^3\pi}{4}$$

3. (C34:3) As per the question, $f(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$. Therefore we can write

$$\phi(x, y) = \int_C \frac{1}{x^2+y^2}(-y, x) \cdot \mathrm{d}(x, y)$$

As suggested in the exercise we will compute the integral along a specific path starting at $(1, 0)$. For given (x, y) we can parameterize the path in two parts with $s_1 : [1, x] \to \mathbb{R}^2$ and $s_2 : [0, y] \to \mathbb{R}^2$. (Here an interval [a,b] is understood as [b,a] if $a > b$.)

$$\begin{aligned}
s_1(t) &= (t, 0) \\
s_2(t) &= (x, t)
\end{aligned}$$

With these notations:

$$\begin{aligned}\phi(x,y) &= \int_C \frac{1}{x^2+y^2}(-y,x)\cdot \mathrm{d}(x,y)\\ &= \int_1^x -\frac{0}{t^2}\mathrm{dt} + \int_0^y \frac{x}{x^2+t^2}\mathrm{d}t = \arctan\frac{y}{x}\end{aligned}$$

Finally, we can check that this is indeed the potential function for $f(x,y)$:

$$\nabla\phi(x,y) = \frac{1}{x^2+y^2}(-y,x) = f(x,y).$$

4. (10.18:13) Note, that the function is not necessarily well defined in $(0,0)$. Thus we will fix our basepoint at $(1,0)$. Then given a point $r(\cos\vartheta,\sin\vartheta)\in\mathbb{R}^2$, then an obvious path from $(1,0)$ to $r(\cos\vartheta,\sin\vartheta)$ can be parametrized by $s_1:[0,\vartheta]\to\mathbb{R}^2$ and $s_2:[1,r]\to\mathbb{R}^2$ with

$$\begin{aligned}s_1(t) &= (\cos t,\sin t)\\ s_2(t) &= t(\cos\vartheta,\sin\vartheta)\end{aligned}$$

For $n\neq -1$

$$\phi(r(\cos\vartheta,\sin\vartheta)) = \int_0^\vartheta a1^n(\cos t,\sin t)\cdot(-\sin t,\cos t)\,dt + \int_1^r at^n(\cos\vartheta,\sin\vartheta)\cdot(\cos\vartheta,\sin\vartheta)\,dt$$

$$= 0 + a\int_1^r t^n\,dt = \frac{ar^{n+1}}{n+1} - \frac{a}{n+1}$$

Checking that it is a potential function:

$$\nabla\frac{ar^{n+1}}{n+1} = ar^n(\cos\vartheta,\sin\vartheta)$$

For $n=-1$ we have

$$\psi(r(\cos\vartheta,\sin\vartheta)) = \int_1^r \frac{a}{t}(\cos\vartheta,\sin\vartheta)\cdot(\cos\vartheta,\sin\vartheta)\mathrm{d}t = a\int_1^r \frac{1}{t}\,dt = a\log r$$

Again checking that this is indeed a potential function:

$$\nabla\psi(r(\cos\vartheta,\sin\vartheta)) = \frac{a}{r}(\cos\vartheta,\sin\vartheta).$$

5. (10.18:17,18) In this exercise

$$f(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

10.18:17 We have computed on the recitation that

$$D_1 f_2(x,y) = D_2 f_1(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2}$$

10.18:18 (Compare the results with 3)

(a) We will consider the 3 cases one by one. First, for $x = 0$ we have, by definition, $\theta = \pi/2$. Now when $x \neq 0$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}.$$

and

$$\arctan\frac{y}{x} = \arctan\frac{-y}{-x} = \phi \in (-\pi/2, \pi/2).$$

For $x > 0$, $-\pi/2 < \theta = \phi < \pi/2$ and this corresponds directly with the definition of the arctan function.
For $x < 0$, it turns out that $\theta = \phi + \pi$ because the angle between (x, y) and $(-x, -y)$ is precisely π.

(b) Using the derivation rule for the inverse function. If $x > 0$

$$\begin{aligned}
\frac{\partial\theta}{\partial x}(x,y) &= \frac{\partial}{\partial x}\arctan\frac{y}{x} \\
&= -\frac{y}{x^2}\frac{1}{1+(\frac{y}{x})^2} = -\frac{y}{x^2+y^2} \\
\frac{\partial\theta}{\partial y}(x,y) &= \frac{\partial}{\partial y}\arctan\frac{y}{x} \\
&= \frac{1}{x}\frac{1}{1+(\frac{y}{x})^2} = \frac{x}{x^2+y^2}
\end{aligned}$$

Similar argument works for $x < 0$ case. For $x = 0$ one computes the left and right derivatives, and see that they are both equal to:

$$\frac{\partial\theta}{\partial x}(0,y) = -\frac{1}{y}$$

and

$$\frac{\partial \theta}{\partial y}(0, y) = 0.$$

Hence for all (x, y), the relations in the exercise for $\frac{\partial \theta}{\partial x}$ and $\frac{\partial \theta}{\partial y}$ hold. This proves that θ is a potential function for f on the set T.

18.024 Multivariable Calculus with Theory
Spring 2011