Solutions for PSet 1

1. (1.10:22)

   (a) Let $S = \{x_1, \ldots, x_k\} \subset V$. As $L(S) = span(S)$, we can write:

   $$L(S) = \{y : y \in V \text{ where } y = \sum_{i=1}^{k} c_i x_i\} \; c_i \text{ is scalar}$$

   For $c_j = 1, c_i = 0, i \neq j$, we have $y = \sum_i c_i x_i = x_j \in L(S)$. Thus $x_j \in S$ implies that $x_j \in L(S)$ and $S \subseteq L(S)$.

   (b) As T is a subspace of linear space V, T is a non-empty subset of V and T satisfies all closure axioms. Since $S \subseteq T$, we know (using the notation above) that $\{x_1, \ldots, x_k\} \subseteq T$. Now let $y \in L(S)$. Then by definition there exist $c_i \in \mathbb{R}$, for $i = 1, \ldots, k$, such that $y = \sum_i c_i x_i$. By the closure axioms, $\sum_i c_i x_i \in T$ and thus $L(S) \subseteq T$.

   (c) Since $L(S)$ is a subspace of V, one direction is obvious.

   Now, suppose by contradiction that S is a subspace of V but $S \neq L(S)$. Since $S \subset L(S)$, this implies there exists $y \in L(S) - S$. As $y \in L(S)$, there exist $c_i \in \mathbb{R}$, $i = 1, \ldots, k$, such that $y = \sum_i c_i x_i$. As S is a subset and thus closed under addition and scalar multiplication, $y \in S$. This implies a contradiction and proves the result.

   (d) Assume $S = \{x_1, \ldots, x_k\}, T = \{x_1, \ldots, x_n\}$ where $n \geq k$. Let $y \in L(S)$. Then $y = \sum_{i=1}^{k} c_i x_i$ for some $c_i \in \mathbb{R}$. For $c_j = 0$ for all $j = k+1, \ldots, n$, $y = \sum_{i=1}^{k} c_i x_i + \sum_{j=k+1}^{n} c_j x_j$. Thus, $y \in L(T)$.

   (e) As S and T are subspaces of V, they are both closed under addition and scalar multiplication. Let $x, y \in S \cap T$ and $c \in \mathbb{R}$. As $cx + y \in S$ and $cx + y \in T$ we see $cx + y \in S \cap T$. Thus $S \cap T$ is closed under addition and multiplication. Therefore $S \cap T$ is a subspace of V.

   (f) Assume $S = \{x_1, \ldots, x_k\}, T = \{y_1, \ldots, y_n\}$. Let $z \in L(S \cap T)$. Then there exist $c_j \in \mathbb{R}$ and $z_j \in S \cap T$ such that $z = \sum_j c_j z_j$. Since $z_j \in S \cap T$, $\sum_j c_j z_j \in L(S), L(T)$. Thus, $z \in L(S) \cap L(T)$.

   (g) Let $S = \{\mathbf{v}_1, \mathbf{v}_2\}, T = \{\mathbf{v}_3, \mathbf{v}_4\}$ where $\mathbf{v}_i \in \mathbb{R}^3$ are each vectors such that $\mathbf{v}_3, \mathbf{v}_4 \notin L(S)$ and $\mathbf{v}_1, \mathbf{v}_2 \notin L(T)$. We can further choose these vectors such that $L(S)$ and $L(T)$ are both planes in $\mathbb{R}^3$ by making sure each pair of vectors is linearly independent. By construction, $S \cap T = \emptyset$ but $L(S) \cap L(T)$ is a line in $\mathbb{R}^3$. So $L(S) \cap L(T) \neq L(S \cap T)$.

2. (1.13:11) In the linear space of all real polynomials, define $(f,g) = \int_0^\infty e^{-t} f(t)g(t)\,dt$.

(a) Let f, g be polynomials. Then $fg = \sum_{i=0}^n a_i x^i$ for some $n \in \mathbb{N}$, $a_i \in \mathbb{R}$. By definition,

$$(f,g) = \int_0^\infty \sum_{i=0}^n e^{-t} a_i t^i \, dt.$$

Using integration by parts, we see that for any fixed $n \in \mathbb{N}$,

$$\int_0^\infty t^n e^{-t}\,dt = -t^n e^{-t}\Big|_0^\infty + \int_0^\infty n t^{n-1} e^{-t}\,dt = \int_0^\infty n t^{n-1} e^{-t}\,dt.$$

Iteratively integrating by parts n times, we see

$$\int_0^\infty t^n e^{-t}\,dt = n! \int_0^\infty e^{-t}\,dt = n!.$$

(To be truly thorough, one should prove this by induction but we leave that to you!)
Thus, for $fg = \sum_{i=0}^n a_i x^i$,

$$(f,g) = \sum_{i=0}^n i! a_i < \infty.$$

(b)

$$\begin{aligned}
(x_n, x_m) &= \int_0^\infty e^{-t} t^n t^m \,dt = \int_0^\infty e^{-t} t^{m+n}\,dt \\
&= (m+n) \int_0^\infty e^{-t} t^{m+n-1}\,dt \\
&= (m+n)(m+n-1) \int_0^\infty e^{-t} t^{m+n-2}\,dt \text{ (by iteratively integrating by parts)} \\
&= (m+n)(m+n-1)\cdots 1 \left[\int_0^\infty e^{-t}\,dt\right] \\
&= (m+n)(m+n-1)\cdot 1 \cdot 1 = (m+n)!
\end{aligned}$$

(c) If $g(t)$ orthogonal to $f(t)$, then:

$$\begin{aligned}(f,g) &= \int_0^\infty e^{-t}(a+bt)(1+t)\,dt \\ &= \int_0^\infty ae^{-t}\,dt + (a+b)\int_0^\infty te^{-t}\,dt + b\int_0^\infty t^2e^{-t}\,dt = 0 \\ \implies & a+a+b+2b = 2a+3b = 0\end{aligned}$$

This means that $2a = -3b$ or polynomials $g(t) = a\left(1 - \frac{2}{3}t\right)$ satisfy the requirement of orthogonality to $f(t) = 1+t$.

3. (2.4:29) Let V denote the linear space of all real functions continuous on the interval $[-\pi, \pi]$. Let S be that subset of V consisting of all f satisfying:

$$\int_{-\pi}^{\pi} f(t)\,dt = \int_{-\pi}^{\pi} f(t)\cos t\,dt = \int_{-\pi}^{\pi} f(t)\sin t\,dt.$$

(a) By definition, $S \subseteq V$. As integration is a linear operation, it can be shown that for $f_1, f_2 \in S$ and $a \in \mathbb{R}$,

$$\int_{-\pi}^{\pi} f_1(t)+f_2(t)\,dt = \int_{-\pi}^{\pi}(f_1(t)+f_2(t))\cos t\,dt = \int_{-\pi}^{\pi}(f_1(t)+f_2(t))\sin t\,dt$$

and

$$\int_{-\pi}^{\pi} af(t)\,dt = \int_{-\pi}^{\pi} af(t)\cos t\,dt = \int_{-\pi}^{\pi} af(t)\sin t\,dt.$$

Thus, S is closed under addition and scalar multiplication.

(b) S contains the functions $f(x)$ defined above if those functions are real and are a part of V. Thus we have to show that $f(x) = \cos(nx)$ and $f(x) = \sin(nx)$ satisfy the integral equations defining V. Start with $f(x) = \cos(nx)$:

$$\begin{aligned}\int_{-\pi}^{\pi}\cos(nt)\,dt &= \frac{1}{n}[sin(\pi) - sin(-\pi)] = 0 \\ \int_{-\pi}^{\pi}\cos(nt)\cos(t)\,dt &= 0.5\int_{-\pi}^{\pi}[\cos(nt+t)+\cos(nt-t)]\,dt = 0 \\ \int_{-\pi}^{\pi}\cos(nt)\sin(t)\,dt &= 0.5\int_{-\pi}^{\pi}[\sin(nt+t)-\sin(nt-t)]\,dt \\ &= 0.5\cos(-(n+1)\pi) - 0.5\cos((n+1)\pi) \\ &\quad +0.5\cos((n-1)\pi) - 0.5\cos(-(n-1)\pi) = 0 \\ &\quad \text{(for both even and odd n)}\end{aligned}$$

A similar derivation makes the case for $f(x) = \sin(nx)$.

(c) S is infinite dimensional if its basis has an infinite number of independent elements. We can prove it is infinite dimensional by proving it is not finite dimensional. As $f_n(x) = \cos(nx)$, $f_n(x)$ is orthogonal to $f_m(x)$ for all $n > 2 \neq m > 2$. Therefore there is no finite basis set of independent elements that can span S.

(d) Using trigonometric identities, observe that for $g(x) \in T(V)$ one has

$$g(x) = \int_{-\pi}^{\pi} f(t)dt + \cos(x)\int_{-\pi}^{\pi} \cos(t)f(t)dt + \sin(x)\int_{-\pi}^{\pi} \sin(t)f(t)dt.$$

Thus, $T(V)$ is three dimensional with basis $\{1, \cos(x), \sin(x)\}$. (Note that since $f \in V$, the three integrals are all elements of $\mathbb{R}$.)

(e) Based on the identity shown in the previous part of the problem, $g = T(f) = 0$ if and only if the three integrals are all zero. Thus, $N(T)$ is precisely equal to the subspace S.

(f) Using the hint, observe that $f(x) = c_1 + c_2 \cos x + c_3 \sin x$ for some $c_1, c_2, c_3 \in \mathbb{R}$. Now evaluating the three integrals that describe $T(f)$ we observe

$$T(f) = cf(x) = 2\pi c_1 + \pi c_2 \cos x + \pi c_3 \sin x.$$

Thus, if $c_1 = 0$ then $f(x) = c_2 \cos x + c_3 \sin x$ and $c = \pi$ (here c_2, c_3 are arbitrary real numbers). If $c_1 \neq 0$, then $c_2 = c_3 = 0$, $f(x)$ is a constant function and $c = 2\pi$.

(g) Let

$$f_j(x) = \begin{cases} 1 & \text{if } x \in [-1/j, 1/j] \\ 0 & \text{otherwise} \end{cases}$$

Then $f_j \to 0$ strongly in L^2 as

$$\int_{\mathbb{R}} |f_j - 0|^2 dx = 4/j^2 \to 0.$$

Observe, however, that $f_j \to f_\infty$ pointwise, where

$$f_\infty(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Notice here that f_j actually also converges strongly to f_∞ in L^2, so though we've found a solution it might not be fully satisfying. A harder question to solve would be the following: Find a sequence of functions f_j such that NO subsequence of f_j converges pointwise to a function but f_j still converges strongly in L^2 to a function.