Exam 1 Solutions

Problem 1. (10 points) Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(1,0,0) = (2,1,4), T(0,1,0) = (4,3,6), T(0,0,1) = (0,-1,2)$.

1. Determine the null space of T.
2. If A is the plane formed by $span(\{(2,5,-3),(-1,-1,1)\})$, write $T(A)$ in parametric form.

$\boxed{\text{Solution}}$ To determine the null space of T, we need to find all vectors $\mathbf{v}$ such that $T\mathbf{v} = \mathbf{0}$. This is equivalent to solving a system of equations. Note that the matrix representation of T is

$$\begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & -1 \\ 4 & 6 & 2 \end{pmatrix}.$$

To solve the system, we row reduce the augmented matrix

$$\begin{pmatrix} 2 & 4 & 0 & | & 0 \\ 1 & 3 & -1 & | & 0 \\ 4 & 6 & 2 & | & 0 \end{pmatrix}.$$

This process gives

$$\begin{pmatrix} 1 & 2 & 0 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 2 & -2 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

It follows that solutions are of the form $v_1 + 2v_3 = 0$ and $v_2 - v_3 = 0$. That is,

$$N(T) = \{\mathbf{v} \in \mathbb{R}^3 \mid \mathbf{v} = t(-2,1,1) \text{ for } t \in \mathbb{R}\}.$$

Now, to find $T(A)$ we need to determine $span\{T(2,5,-3), T(-1,-1,1)\}$. Matrix multiplication immediately gives

$$T(2,5,-3) = (24,20,32); \qquad T(-1,-1,1) = (-6,-5,-8).$$

Notice that both of these vectors are multiples of $(6,5,8)$. Thus, $T(A)$ is a line through the origin spanned by that vector. In parametric form we have

$$T(A) = \{\mathbf{v} \in \mathbb{R}^3 \mid \mathbf{v} = t(6,5,8) \text{ for } t \in \mathbb{R}\}.$$

Problem 2. (10 points) Let

$$F(t) = \begin{cases} (\sin t, -\cos t) & t \in [0, \pi] \\ (\sin t, \cos t + 2) & t \in (\pi, 2\pi] \end{cases}$$

1. Find $F'(\pi)$, if it is well defined.
2. Find $F''(\pi)$, if it is well defined.
3. Determine $\kappa(t)$ everywhere it is defined.

Solution Away from $t = 0, \pi, 2\pi$ F has first and second derivatives in t. Notice that

$$F'(t) = \begin{cases} (\cos t, \sin t) & t \in (0, \pi) \\ (\cos t, -\sin t) & t \in (\pi, 2\pi) \end{cases}$$

and

$$F''(t) = \begin{cases} (-\sin t, \cos t) & t \in (0, \pi) \\ (-\sin t, -\cos t) & t \in (\pi, 2\pi) \end{cases}.$$

I want to highlight here that many of you wrote something like what was above but with closed brackets. Remember the derivative definition requires a left and right hand limit!

Now,

$$\lim_{t \to \pi^+} F'(t) = (\cos \pi, -\sin \pi) = (-1, 0) = (\cos \pi, \sin \pi) = \lim_{t \to \pi^-} F'(t).$$

Thus, $F'(\pi) = (-1, 0)$. Also,

$$\lim_{t \to \pi^+} F''(t) = (-\sin \pi, -\cos \pi) = (0, 1) \neq (0, -1) = (-\sin \pi, \cos \pi) = \lim_{t \to \pi^-} F''(t).$$

Therefore, F'' is not defined at $t = \pi$.

The final part of this problem can be easily solved if you notice that F is carving out two portions of two different circles of radius equal to one. Thus $\kappa(t) = 1$ everywhere it is defined.

Problem 3: (10 points) Let $f(x, y, z) = x^2 + y^2 + z^2$. Prove f is differentiable at $(1, 1, 1)$ with linear transformation $T(x, y, z) = 2x + 2y + 2z$.

Solution To prove f is differentiable with total derivative T as described we need to show

$$\lim_{||\mathbf{v}||\to 0} \frac{f(\mathbf{v}+(1,1,1)) - f(1,1,1) - T(\mathbf{v})}{||\mathbf{v}||} = 0.$$

Now observe that

$$f(\mathbf{v}+(1,1,1))-f(1,1,1)-T(\mathbf{v}) = (v_1+1)^2+(v_2+1)^2+(v_3+1)^2-3-2v_1-2v_2-2v_3 = v_1^2+v_2^2+v_3^2.$$

Thus

$$\lim_{||\mathbf{v}||\to 0} \frac{f(\mathbf{v}+(1,1,1)) - f(1,1,1) - T(\mathbf{v})}{||\mathbf{v}||} = \lim_{||\mathbf{v}||\to 0} \frac{||\mathbf{v}||^2}{||\mathbf{v}||} = \lim_{||\mathbf{v}||\to 0} ||\mathbf{v}|| = 0.$$

It follows that f is differentiable at $(1,1,1)$ with the total derivative as described.

Problem 4. (15 points) Consider the set $\mathcal{L}(\mathbb{R}^3,\mathbb{R}^2)$ of all linear maps L from $\mathbb{R}^3$ to $\mathbb{R}^2$ and define addition of $L, K \in \mathcal{L}(\mathbb{R}^3,\mathbb{R}^2)$ the following way:

$$(L+K)(v) = L(v) + K(v) \qquad (v \in \mathbb{R}^3)$$

Define multiplication by a constant c as:

$$(cL)(v) = c(L(v)) \qquad (v \in \mathbb{R}^3)$$

1. Are the linear maps $L(x,y,z) = (x,0)$, $K(x,y,z) = (y,0)$, $N(x,y,z) = (x,y)$ linearly independent? Prove it either way.
2. Find a basis for $\mathcal{L}(\mathbb{R}^3,\mathbb{R}^2)$.
3. What is the dimension of $\mathcal{L}(\mathbb{R}^3,\mathbb{R}^2)$?

Solution

1. The given maps are linearly independent. Here is why. Suppose $c_1L + c_2K + c_3N = \mathbf{0}$ where here $\mathbf{0}$ is the zero transformation. That is, $\mathbf{0}(x,y,z) = (0,0)$ for all $(x,y,z) \in \mathbb{R}^3$. Then $(c_1L+c_2K+c_3N)(x,y,z) = (c_1x+c_2y+c_3x, c_3y)$. This implies $c_3 = 0$ and thus $c_1x + c_2y = 0$ for all $x, y, \in \mathbb{R}$. Therefore, $c_1 = c_2 = 0$ as well. Thus, the only linear combination of the three maps that gives the zero map has all coefficients equal to zero.

2. A good basis can be given by the 6 functions $L_1^1(x,y,z) = (x,0)$, $L_1^2(x,y,z) = (0,x)$, $L_2^1(x,y,z) = (y,0)$, $L_2^2(x,y,z) = (0,y)$, $L_3^1(x,y,z) = (z,0)$ and $L_3^2(x,y,z) = (0,z)$.

   To check that these maps are linearly independent suppose, that:

$$\alpha_1^1 L_1^1 + \alpha_1^2 L_1^2 + \alpha_2^1 L_2^1 + \alpha_2^2 L_2^2 + \alpha_3^1 L_3^1 + \alpha_3^2 L_3^2 = 0$$

   for some numbers α_i^j $(1 \leq i \leq 3,\ 1 \leq j \leq 2)$. We would like to prove that all α_i^j's are equal to 0. Remember, that 0 in this vector space was the function defined as $0(x,y,z) = (0,0)$. Then the above equation translates to

$$(\alpha_1^1 L_1^1 + \alpha_1^2 L_1^2 + \alpha_2^1 L_2^1 + \alpha_2^2 L_2^2 + \alpha_3^1 L_3^1 + \alpha_3^2 L_3^2)(x,y,z) = (\alpha_1^1 x + \alpha_2^1 y + \alpha_3^1 z, \alpha_1^2 x + \alpha_2^2 y + \alpha_3^2 z) = (0,0)$$

   for every $(x,y,z) \in \mathbb{R}^3$. Substituting $(x,y,z) = (1,0,0)$ to the above equation gives:

$$(\alpha_1^1, \alpha_1^2) = (0,0)$$

   which means that $\alpha_1^1 = \alpha_2^1 = 0$. Similarly substituting $(x,y,z) = (0,1,0)$ gives $\alpha_2^1 = \alpha_2^2 = 0$, finally $(x,y,z) = (0,0,1)$ gives $\alpha_3^1 = \alpha_3^2 = 0$. This proves that the linear maps L_i^j $(1 \leq i \leq 3,\ 1 \leq j \leq 2)$ were linearly independent.

   To see that the maps L_i^j $(1 \leq i \leq 3,\ 1 \leq j \leq 2)$ also generate the vector space of linear maps take an arbitrary linear map $K : \mathbb{R}^3 \to \mathbb{R}^2$. Let us denote the projections from $\mathbb{R}^2 \to \mathbb{R}^1$ to the first coordinate by π_1 and to the second coordinate by π_2. Thus $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$. Now $K(x,y,z) \in \mathbb{R}^2$, thus the terms $\pi_1(K(x,y,z)) \in \mathbb{R}$ and $\pi_2(K(x,y,z)) \in \mathbb{R}$ are the first and second coordinates of $K(x,y,z)$, respectively. Consider the linear function:

$$\begin{aligned} L &= \pi_1(K(1,0,0))L_1^1 + \pi_2(K(1,0,0))L_1^2 + \\ &+ \pi_1(K(0,1,0))L_2^1 + \pi_2(K(0,1,0))L_2^2 + \\ &+ \pi_1(K(0,0,1))L_3^1 + \pi_2(K(0,0,1))L_3^2 \end{aligned}$$

   then

$$\begin{aligned} L(x,y,z) &= \big(\pi_1(K(1,0,0))L_1^1 + \pi_2(K(1,0,0))L_1^2 + \\ &\quad + \pi_1(K(0,1,0))L_2^1 + \pi_2(K(0,1,0))L_2^2 + \\ &\quad + \pi_1(K(0,0,1))L_3^1 + \pi_2(K(0,0,1))L_3^2\big)(x,y,z) \\ &= (\pi_1(K(1,0,0)x + K(0,1,0)y + K(0,0,1)z), \\ &\quad \pi_2(K(1,0,0)x + K(0,1,0)y + K(0,0,1)z)) \\ &= (\pi_1(K(x,y,z)), \pi_2(K(x,y,z))) = K(x,y,z). \end{aligned}$$

Thus we could express any linear map K in terms of L_i^j $(1 \leq i \leq 3, 1 \leq j \leq 2)$. So they indeed span the space.

3. There is a 6 element basis, thus the dimension is 6.

Problem 5. (15 points) Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ that satisfies the following conditions:

1. For all fixed $x_0 \in \mathbb{R}$ the function $f_{x_0} = f(x_0, y)\colon \mathbb{R} \to \mathbb{R}$ is continuous and;
2. For all fixed $y_0 \in \mathbb{R}$ the function $f^{y_0} = f(x, y_0)\colon \mathbb{R} \to \mathbb{R}$ is continuous and;
3. For all fixed $x_0 \in \mathbb{R}$ the function f_{x_0} is monotonically increasing in y, i.e. if $y > y'$ then, $f(x_0, y) > f(x_0, y')$.

Prove f is continuous.

Solution This solution will appear later. You'll have another chance to work on it.