EXAM 2 - APRIL 15, 2011

Name:

NOTE: If at any point during a calculation you are using a theorem from class, justify the calculation by stating the appropriate theorem.

(1) (10 points) Consider $f(x,y) = (xy + y)^{10}$ on the square $Q = [0,1] \times [0,1]$. Evaluate $\int\int_Q f dxdy$.

(2) (5 points) Complete the following statement. (There is more than one correct answer.)
Let $S \subset \mathbb{R}^n$ be open and connected. Suppose $\mathbf{f}$ is a vector field defined on S. Then $\mathbf{f}$ is a gradient field if and only if ——————.

(3) (10 points) Let γ be the semi-circle connecting $(0,0)$ and $(2,0)$ that sits in the half plane where $y \geq 0$. Given $\mathbf{f}(x,y) = (2x + \cos y, -x \sin y + y^7)$, calculate $\int \mathbf{f} \cdot d\gamma$.

(4) Consider the surface $x^2yz + 2xz^2 = 6$ in $\mathbb{R}^3$.

(a) (3 points) For $(x, y) = (1, 4)$, determine all values of z such that $(1, 4, z)$ is on the surface.

(b) (6 points) For each of the values of z found above, determine at which of the points $(1, 4, z)$ one can find a neighborhood on the surface and a function $g : \mathbb{R}^2 \to \mathbb{R}$ such that the neighborhood can be described by the points $(x, y, g(x, y))$.

(c) (6 points) Choose one point from part (b) where the implicit function theorem can be applied and let $g(x, y) = z$ be the function defined in a neighborhood of $(1, 4)$ such that $(x, y, g(x, y))$ is on the surface. Find $\nabla g(1, 4)$.

(5) (15 points) Assuming the comparison theorem for step functions, prove it for integrable functions $f, g : U \to \mathbb{R}$. That is, let U be a closed rectangle in $\mathbb{R}^3$ and assume $\int\int_U f, \int\int_U g$ both exist. If $g \leq f$ for all $\mathbf{x} \in U$, prove $\int\int_U g \leq \int\int_U f$.

BONUS: (6 points)

(a) Let A be a set of content zero and assume $B \subset A$. Prove B has content zero.
(b) Let A_i, $i = 1, \ldots, n$ be sets of content zero. Prove $\cup_{i=1}^{n} A_i$ has content zero.
(c) Provide a counterexample to the following statement (and explain it): Let $\{A_i\}_{i=1}^{\infty}$ be a collection of sets A_i which each have content zero. Then $\cup_{i=1}^{\infty} A_i$ has content zero.