Exam 2 Solutions

Problem 1. Consider $f(x,y) = (xy+y)^{10}$ on the square $Q = [0,1]\times[0,1]$. Evaluate $\int\int_Q f dxdy$.

Solution The function

$$f(x,y) = (xy+y)^{10}$$

is continous on $\mathbb{R}^2$. Thus, $\int_0^1 f(x,y)dx$ is integrable for all $y \in [0,1]$ so one can apply Fubini's Theorem to get:

$$\int\int_Q (xy+y)^{10}\,dx\,dy = \int_0^1\int_0^1 (xy+y)^{10}\,dx\,dy = \int_0^1 \left[\frac{(xy+y)^{11}}{11y}\right]_0^1 dy = \int_0^1 \frac{(2^{11}-1)y^{10}}{11}dy = \frac{2047}{121}$$

Problem 2. Complete the following statement. (There is more than one correct answer.)
Let $S \subset \mathbb{R}^n$ be open and connected. Suppose $\mathbf{f}$ is a vector field defined on S. Then $\mathbf{f}$ is a gradient field if and only if ——.

Solution There are two correct answers:

> ...The line integral of f along a path connecting two points $\mathbf{a}, \mathbf{b} \in S$ is independent of the path in S;
>
> ...The line integral of f is 0 around every piecwise smooth closed path in S.

Problem 3. Let γ be the semi-circle connecting $(0,0)$ and $(2,0)$ that sits in the half plane where $y \geq 0$. Given $\mathbf{f}(x,y) = (2x+\cos y, -x\sin y + y^7)$, calculate $\int \mathbf{f}\cdot d\gamma$. If your calculation requires justification from a theorem we proved in class, state the theorem you are using.

Solution Notice that $D_1 f_2(x,y) = -\sin y = D_2 f_1(x,y)$. Since $f(x,y)$ is defined on all of $\mathbb{R}^2$, which is convex, we conclude $f(x,y)$ is a gradient field. Thus the integral of $f(x,y)$ from $(0,0)$ to $(2,0)$ is independent of the path. Let us integrate on a straight line $s : [0,2] \to \mathbb{R}^2$ defined by $s(t) = (t,0)$:

$$\int_C (2x+\cos y, -x\sin y+y^7)\,ds = \int_0^2 (2t+\cos 0, -t\sin 0+0^7)\cdot(1,0)\,dt = \left[t^2+t\right]_0^2 = 6$$

Problem 4. Consider the surface $x^2yz + 2xz^2 = 6$ in $\mathbb{R}^3$.

1. For $(x, y) = (1, 4)$, determine all values of z such that $(1, 4, z)$ is on the surface.

2. For each of the values of z found above, determine at which of the points $(1, 4, z)$ one can apply the implicit function theorem.

3. Choose one point from part (b) where the implicit function theorem can be applied and let $g(x, y) = z$ be the function defined in a neighborhood of $(1, 4)$ such that $(x, y, g(x, y))$ is on the surface. Find $\nabla g(1, 4)$.

Solution

1. By substitution we get $4z + 2z^2 = 6$ or $z^2 + 2z - 3 = 0$. This easily factors into $(z - 1)(z + 3) = 0$ and the zero product property implies $z = 1$ or $z = -3$.

2. Let $f(x, y, z) = x^2yz + 2xz^2 - 6$. Then $\frac{\partial f}{\partial z} = x^2y + 4xz$. And thus $\frac{\partial f}{\partial z}(1, 4, 1) = 4+4 = 8 \neq 0$, $\frac{\partial f}{\partial z}(1, 4, -3) = 4-12 = -8 \neq 0$. Thus, one can apply the implicit function theorem at both points, as the only necessary condition $(\partial f/\partial z \neq 0)$ has been met.

3. By the implicit function theorem we know that

$$\nabla g = -\left(\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}, \frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}\right).$$

   Since $\frac{\partial f}{\partial x} = 2xyz + 2z^2$, $\frac{\partial f}{\partial y} = x^2z$, wherever g is defined we see

$$\nabla g(x, y) = \left(-\frac{2xyz + 2z^2}{x^2y + 4xz}, -\frac{x^2z}{x^2y + 4xz}\right)$$

   and thus if we defined g in a neighborhood of $(1, 4, 1)$, $\nabla g(1, 4) = \left(-\frac{10}{8}, -\frac{1}{8}\right) = (-5/4, -1/8)$. While if g is defined in a neighborhood of $(1, 4, -3)$, $\nabla g(1, 4) = \left(-\frac{-6}{-8}, -\frac{-3}{-8}\right) = (-3/4, -3/8)$.

Problem 5. Assuming the comparison theorem for step functions, prove it for integrable functions $f, g : U \to \mathbb{R}$. That is, let U be a closed rectangle in $\mathbb{R}^3$ and assume $\int\int_U f, \int\int_U g$ both exist. If $g \leq f$ for all $\mathbf{x} \in U$, prove $\int\int_U g \leq \int\int_U f$.

Solution We proceed by contradiction. Assume f, g are both integrable in U and $g \leq f$ in U but $\int\int_U g > \int\int_U f$. Let $M = \int\int_U (g - f)$. By hypothesis, $M > 0$. The Riemann condition implies there exist step functions s_g, t_g, s_f, t_f with

- $s_g \leq g \leq t_g$ and $s_f \leq f \leq t_f$ in U,
- $\int\int_U t_g - s_g < M/4, \int\int_U t_f - s_f < M/4$, and
- $\int\int_u s_g \leq \int\int_U g \leq \int\int_U t_g$, $\int\int_U s_f \leq \int\int_U f \leq \int\int_U t_f$.

Taken together, these imply that $\int\int_U g - \int\int_U s_g < M/4, \int\int_U f - \int\int_U t_f > -M/4$.

Now consider

$$\int\int_U s_g - \int\int_U t_f > (\int\int_U g - M/4) + (-\int\int_U f - M/4) = \int\int_U g - \int\int_U f - M/2 = M - M/2 > 0.$$

But since by construction $s_g \leq t_f$ on U, the comparison theorem for step functions implies that $\int\int_U t_f \geq \int\int_U s_g$. This gives the necessary contradiction.

Bonus.

1. Let A be a set of content zero and assume $B \subset A$. Prove B has content zero.
2. Let A_i, $i = 1, \ldots, n$ be sets of content zero. Prove $\cup_{i=1}^n A_i$ has content zero.
3. Provide a counterexample to the following statement (and explain it): Let $\{A_i\}_{i=1}^\infty$ be a collection of sets A_i which each have content zero. Then $\cup_{i=1}^\infty A_i$ has content zero.

Solution

1. For any $\epsilon > 0$, let R_i be a finite collection of rectangles such that $A \subset \cup_i R_i$ and $\sum_i Area(R_i) < \epsilon$. But $B \subset A$ implies $B \subset \cup_i R_i$.
2. For any $\epsilon > 0$, let R_j^i, $j = 1, \ldots, m_i$, be a finite collection of rectangles such that $A_i \subset \cup_{j=1}^{m_i} R_j^i$ and $\sum_{j=1}^{m_i} Area(R_j^i) < \epsilon/n$. But then $\cup_{i=1}^n A_i \subset \cup_{i=1}^n \cup_{j=1}^{m_i} R_j^i$ and $\sum_{i=1}^n \sum_{j=1}^{m_i} Area(R_j^i) < \epsilon$. Thus, $\cup_{i=1}^n A_i$ has content zero.
3. Index the rational numbers between $[0, 1]$ by i, and represent each element in the set by r_i. Let A_i represent the line segment connecting $x = 0$ and $x = 1$ at height $y = r_i$. Each A_i certainly has content zero as each is a function $y = f(x)$, continuous on $[0, 1]$. But $\cup_{i=1}^\infty A_i$ is dense in the rectangle $[0, 1] \times [0, 1]$.

   Now, fix $\epsilon = 1/2$ and suppose there exists a finite collection of rectangles R_j, $j = 1, \ldots, m$ such that $\cup_{i=1}^\infty A_i \subset \cup_{j=1}^m R_j$ with $\sum_{j=1}^m Area(R_j) < 1/2$. Since

$Area([0,1]\times[0,1]) = 1$, there exists some open ball $B \subset [0,1]\times[0,1]$ such that $B \cap \cup_{j=1}^{m} R_j = \emptyset$. Let x denote the center of the ball B. Notice that there exists some $r_k \in \mathbb{Q}$ such that $(x, r_k) \in B$. (In fact, there are an infinite number of such r_k.) Since $(x, r_k) \in A_k$ we get a contradiction. That is, $\cup_{i=1}^{\infty} A_i$ is not contained in any finite collection of rectangles with area less than $1/2$.

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